shown in §§80 and 97, entirely analogous results are possible for radiation in the limit that $\lambda_p/l \ll 1$, where λ_p is the photon mean free path.

The real problem arises near boundary surfaces where a mean free path (photon or particle) may exceed any characteristic structural length in the flow. We must then find other methods for evaluating the averages that appear as the energy flux or as nonisotropic (perhaps even off-diagonal) contributions to the stress tensor, in the fluid and/or radiation energy and momentum equations. We have ignored this problem for ordinary fluids because it becomes important only in extremely rarefied flows [e.g., the interplanetary medium (**H3**)]. But it cannot be ignored for radiation because we always must deal with regions in which $\lambda/l \ll 1$ while $\lambda_p/l \gg 1$; indeed these are the very layers of a radiating flow that we can observe. Here we must face the closure problem squarely.

In one-dimensional problems we have two equations containing the three scalars E_ν, F_ν, and P_ν, and one approach is to close the system with variable Eddington factors f_ν, as in (78.16b) and (78.19b). When solving the moment equations we assume that f_ν is known. We subsequently determine f_ν from a separate angle-by-angle formal solution of the full transfer equation assuming that the radiation energy density (which appears in the source function) is known; we then iterate the two steps to convergence. As the value of f_ν converges, the closure becomes essentially exact. In radiation-hydrodynamics calculations where computational speed is paramount, a yet-simpler procedure is sometimes adopted: one uses approximate analytical formulae to determine f from the geometry of the problem and from that ratio (F/E) [see, e.g., (**F2**), (**S4**)]. G. Minerbo (**M6**) developed an elegant formulation of this kind; Minerbo's formulation is useful also in multidimensional problems where the full Eddington tensor $\mathsf{f} \equiv \mathsf{P}/E$ must be specified.

Alternatively, we can rewrite the transfer equation in terms of angle-dependent mean-intensity-like and flux-like variables (see §83), and obtain exact closure of two coupled angle-dependent equations that strongly resemble the moment equations, and have many of their desirable properties. These equations can be discretized and solved directly by efficient numerical methods.

6.5 Solution of the Transfer Equation

We now address the problem of *solving* the transfer equation. To develop insight we first discuss the formal solution and special solutions for important special cases; we then discuss general numerical techniques. Inasmuch as we now focus mainly on mathematical rather than physical content of the equations, we will usually use the Eddington variables J_ν, H_ν, and K_ν in preference to the dynamical variables E_ν, F_ν, and P_ν.

We concentrate almost entirely on the solution of the *time-independent* transfer equation (the exception is an analytical expression for the time-dependent formal solution). The techniques developed here provide a foundation for later work. Equally important, it turns out that ignoring the time dependence of the transfer eqution itself is actually not a bad approximation for many astrophysical flow problems. To see why this is so, let us examine some characteristic time scales.

To deal with *radiation flow* (e.g., the propagation of a radiation front into material) we must consider very short time scales, $t_R \sim l/c$ or $t_\lambda \sim \lambda_p/c$, corresponding to a photon flight time over a characteristic structural length l, or over a photon mean free path λ_p. For such problems it is obviously necessary to solve the full time-dependent transfer equation in order to describe properly the dynamics of the radiating flow. In Chapter 7 we analyze the relative importance of terms in the transfer equation for moving media in the radiation flow limit, and some of the numerical methods discussed in §7.3 can be used in this regime. Nevertheless, we normally consider such problems to lie outside the scope of this book.

The primary concerns of this book is with problems of *fluid flow* on typical dynamical time scales $t_f \sim l/v$, where l is a characteristic length in the flow, and v is a typical flow velocity; for example, l might be of the order of a scale height H, and v of the order of the material sound speed a. In an optically thin region, the ratio t_R/t_f is $O(v/c)$, which is a small number in the astrophysical flows we will consider; in a stellar pulsation $(v/c) \sim 10^{-4}$, in a stellar wind $(v/c) \sim 10^{-2}$, and even in a supernova explosion $(v/c) \lesssim 3 \times 10^{-2}$. (In some flows, e.g., relativistic collapse or laser-fusion experiments, t_f can be very much shorter, approaching t_R.) When $t_R \ll t_f$, the radiation field at any position adjusts essentially instantaneously to changes in physical conditions throughout the flow. This means that on time scales appropriate to a calculation of flow dynamics we can *ignore* the explicit time variation of the radiation field, and can consider it to be in a sequence of quasi-steady states, each of which is consistent with the instantaneous physical structure of the flow. We describe the radiation field in this regime as *quasi static* or *quasi stationary*. Of course we must still account explicitly for time derivatives of the radiation energy and momentum densities in the energy and momentum conservation equations for the radiating fluid, where these quantities appear on an equal footing with the corresponding material terms (see Chapters 7 and 8). The same remark also applies in the cases now to be discussed.

In optically thick regions (e.g., a stellar interior) the situation is different. Here photons *diffuse* by a random-walk process with a mean free path λ_p, and we can essentially discard the transfer equation, replacing it with a simple asymptotic solution (cf. §80). In the diffusion process a photon suffers of the order of $(l/\lambda_p)^2$ interactions with the material as it travels a distance l; hence the characteristic radiation *diffusion time* is $t_d \sim (l^2/c\lambda_p)$. The ratio (t_d/t_f) is thus $O(lv/\lambda_p c)$. In the *true* or (*static*) *diffusion limit*, the dynamical time scale of interest (e.g., a nuclear-evolution time) is so long that $t_f \gg t_d$ [with implies that $(v/c) \ll (\lambda_p/l)$], and random-walk diffusion sets the response time of the radiation field to changes in physical conditions;

because $(l/\lambda_p) \gg 1$, this response can be quite slow. At first sight it might seem that we should adopt timesteps of the order of t_d in our calculation, even though we are not interested in radiation diffusion phenomena per se, but wish only to follow events on the time scale t_f. However, in this regime the radiation field saturates to its thermal equilibrium value (see §80), with high collision rates assuring rapid photon creation and destruction in the dense material. Hence the radiation field is closely *frozen* to instantaneous local conditions as they vary. Therefore, as we computationally follow the time evolution of the fluid properties, we automatically track the time variation of the radiation field, which simply passes through a sequence of states in instantaneous equilibrium with local physical conditions as determined by the flow. In effect the radiation field is again quasi stationary.

If, on the other hand, the material is optically thick enough to trap photons, but now $(v/c) \gtrsim (\lambda_p/l)$ so that $t_f \lesssim t_d$, flow-induced changes in the physical properties of the medium can *drive* changes in the radiation field faster than they could have occurred if only diffusion were operative. We refer to this regime as the *dynamic diffusion* limit. Here we must account for the coupled time variation of the material properties and radiation field by performing a time-dependent solution of both the momentum and energy equations for the radiating fluid, and the radiation energy and momentum equations, simultaneously, on a fluid-flow time scale.

In general we must use radiation moment equations that bridge the optically thick and thin limits correctly, and we will see in Chapter 7 that it is then essential to include terms that are formally of $O(v/c)$. But, in any event, the only reason we need to solve the transfer equation itself is to obtain Eddington factors, required to close the moment equations. In most problems these ratios, which reflect almost purely *geometric* information about source distributions and boundaries, are given with good accuracy by a static *snapshot*, that is, a solution of the time-independent transfer equation using current values of the physical properties in the flow.

In sum, we need to solve the full time-dependent *transfer* equation (in distinction to *moment* equations) only when we wish to treat radiation flow; but generally not if we wish merely to treat fluid flow. We thus see that there is ample motivation for studying the time-independent transfer problem.

79. *Formal Solution*

BOUNDARY CONDITIONS

The radiation field in any volume consists both of photons emitted by the material within the volume, and of radiation that penetrates the bounding surface of the volume from imposed external sources. The latter are fixed by appropriate *boundary conditions*, which are needed to specify a unique solution. For an arbitrary convex volume V bounded by surface S we must therefore be given

$$I(\mathbf{x}_S, t; \mathbf{n}, \nu) = f(\mathbf{x}_S, t; \mathbf{n}, \nu) \tag{79.1}$$

for all $\mathbf{x}_S$ on S, along all rays $\mathbf{n}$ that penetrate into V (i.e., for $\mathbf{n} \cdot \mathbf{N} \leq 0$, where $\mathbf{N}$ is the outward normal of S), at all frequencies.

In astrophysical problems the generality of (79.1) is usually unnecessary because we consider planar or spherically symmetric geometries. In planar geometry we encounter two classes of problems: (1) a *finite slab* of thickness Z, and (2) a *semi-infinite atmosphere*, which is a medium (such as a stellar atmosphere) that has an open boundary surface separating vacuum from material that is so optically thick that it can be imagined to extend to infinity.

For the finite slab we must specify functions f^+ and f^- describing the incoming radiation on both faces, that is,

$$I(Z, t; \mu, \nu) = f^-(t; \mu, \nu), \qquad \mu \leq 0, \tag{79.2}$$

at the boundary nearest the observer (the *top* of the slab), and

$$I(0, t; \mu, \nu) = f^+(t; \mu, \nu), \qquad \mu \geq 0, \tag{79.3}$$

at the farther boundary (the *bottom* of the slab).

For a semi-infinite atmosphere we have an upper boundary condition of the form (79.2). In posing a lower boundary condition, we note that at very great depth the radiation field must satisfy a *boundedness* condition

$$\lim_{\tau_\nu \to \infty} [e^{-\tau_\nu/\mu} I(\tau_\nu, t; \mu, \nu)] = 0 \tag{79.4}$$

for the solution to be well behaved mathematically. In practice the solution must always be truncated at some large depth, where we impose a boundary condition that expresses f^+ in terms of the local source function and its gradient, or fixes the flux transported across the boundary. These conditions follow naturally from physical considerations in the diffusion limit (cf. §80).

In spherical geometry we deal with *spherical shells* or semi-infinite *spherical envelopes*. For a spherical shell we have equations analogous to (79.2) and (79.3) at $r = R$ (outer radius) and at $r = r_c$ (inner or *core* radius) respectively. In the semi-infinite case, we use (79.2) at $r = R$, and apply the diffusion approximation at $r = r_c$.

For time-dependent problems, the spatial boundary conditions must be augmented by an *initial condition* that gives the radiation field within V at $t = 0$, that is,

$$I(\mathbf{x}, 0; \mathbf{n}, \nu) = g(\mathbf{x}; \mathbf{n}, \nu) \tag{79.5}$$

for all $\mathbf{x}$ within V.

THE TIME-INDEPENDENT FORMAL SOLUTION

Consider now the transfer equation for a static medium (or steady flow), so that $(\partial/\partial t)\equiv 0$. Then in Cartesian coördinates

$$\mathbf{n}\cdot\nabla I=\eta-\chi I \tag{79.6}$$

where all variables are functions of $(\mathbf{x};\mathbf{n},\nu)$. The operator $\mathbf{n}\cdot\nabla=n^i(\partial/\partial x^i)$ is the derivative with respect to path length along the ray whose direction is $\mathbf{n}$. Radiation at $\mathbf{x}$ moving in direction $\mathbf{n}$ consists of photons that were emitted along $\mathbf{n}$ from points along the line $\mathbf{x}-|\mathbf{x}'-\mathbf{x}|\,\mathbf{n}$. It is therefore convenient to let s be the path length *backward* along $\mathbf{n}$, so that (79.6) becomes

$$-\frac{d}{ds}[I(\mathbf{x}-\mathbf{n}s;\mathbf{n},\nu)]+\chi(\mathbf{x}-\mathbf{n}s;\mathbf{n},\nu)I(\mathbf{x}-\mathbf{n}s;\mathbf{n},\nu)=\eta(\mathbf{x}-\mathbf{n}s;\mathbf{n},\nu). \tag{79.7}$$

In what follows we suppress mention of $(\mathbf{n},\nu)$ for brevity.

Equation (79.7) is a linear, first-order differential equation, which has an integrating factor $\exp[-\tau(s_0,s)]$, where

$$\tau(s_0,s)\equiv\int_{s_0}^{s}\chi(\mathbf{x}-\mathbf{n}s'')\,ds'', \tag{79.8}$$

and s_0 is an arbitrary point along $\mathbf{n}$. Notice that τ depends on both $\mathbf{n}$ and ν. Using (79.8) to rewrite (79.7) we have

$$-\frac{d}{ds}[I(\mathbf{x}-\mathbf{n}s)e^{-\tau(s_0,s)}]=e^{-\tau(s_0,s)}\eta(\mathbf{x}-\mathbf{n}s), \tag{79.9}$$

whence

$$I(\mathbf{x}-\mathbf{n}s)e^{-\tau(s_0,s)}-I(\mathbf{x}-\mathbf{n}s_0)=\int_{s}^{s_0}\eta(\mathbf{x}-\mathbf{n}s')e^{-\tau(s_0,s')}\,ds', \tag{79.10}$$

or, finally,

$$I(\mathbf{x}-\mathbf{n}s)=e^{-\tau(s,s_0)}I(\mathbf{x}-\mathbf{n}s_0)+\int_{s}^{s_0}\eta(\mathbf{x}-\mathbf{n}s')e^{-\tau(s,s')}\,ds'. \tag{79.11}$$

To obtain I at a specific $\mathbf{x}$ we set $s=0$, and choose s_0 such that $\mathbf{x}_S=\mathbf{x}-s_0\mathbf{n}$ lies on the boundary surface S. Then

$$\begin{aligned}I(\mathbf{x};\mathbf{n},\nu)=&\int_0^{s_0}\eta(\mathbf{x}-\mathbf{n}s';\mathbf{n},\nu)\exp\left[-\int_0^{s'}\chi(\mathbf{x}-\mathbf{n}s'';\mathbf{n},\nu)\,ds''\right]ds'\\&+f(\mathbf{x}_S;\mathbf{n},\nu)\exp\left[-\int_0^{s_0}\chi(\mathbf{x}-\mathbf{n}s'';\mathbf{n},\nu)\,ds''\right].\end{aligned} \tag{79.12}$$

Equation (79.12) is the general formal solution of the time-independent transfer problem. It states that the intensity of radiation traveling along $\mathbf{n}$ at point $\mathbf{x}$ is the sum of photons emitted from all points along the line segment $\mathbf{x}-\mathbf{n}s$, attenuated by the integrated absorptivity of the intervening material, plus an attenuated contribution from photons entering the boundary surface where it is pierced by that line segment. Note, however, that the apparent simplicity of (79.12) is illusory, for if η contains scattering terms (which depend on I), then (79.12) is an *integral equation* that must be *solved* for I, and therefore brings us no closer to the answer than the original differential equation. But if η is purely thermal, or is given, I can be computed from (79.12) by quadrature; this computation can be done either by direct evaluation of (79.12), or by equivalent, efficient, differential-equation techniques described in §83.

PLANAR MEDIA

Consider (79.12) for a static, planar, semi-infinite medium with no radiation incident at the upper boundary (e.g., a stellar atmosphere). For outgoing radiation at height z in the medium

$$I(z;\mu,\nu)=\int_{-\infty}^{z}\eta(z';\nu)e^{-(\tau_\nu'-\tau_\nu)/\mu}\,dz'/\mu,\qquad(0\leq\mu\leq1),\tag{79.13}$$

or

$$I(\tau_\nu;\mu,\nu)=\int_{\tau_\nu}^{\infty}S_\nu(\tau_\nu')e^{-(\tau_\nu'-\tau_\nu)/\mu}\,d\tau_\nu'/\mu,\qquad(0\leq\mu\leq1),\tag{79.14}$$

where $S_\nu=\eta_\nu/\chi_\nu$ is the source function, and we have invoked (79.4). Similarly, for incoming radiation

$$I(\tau_\nu;\mu,\nu)=\int_{0}^{\tau_\nu}S_\nu(\tau_\nu')e^{(\tau_\nu-\tau_\nu')/\mu}\,d\tau_\nu'/(-\mu),\qquad(-1\leq\mu\leq0).\tag{79.15}$$

Equation (79.14) yields directly the *emergent intensity* seen by an observer outside the atmosphere ($\tau_\nu=0$):

$$I(0;\mu,\nu)=\int_0^\infty S_\nu(\tau_\nu)e^{-\tau_\nu/\mu}\,d\tau_\nu/\mu.\tag{79.16}$$

If we assume that near the surface S_ν is given by the linear expansion $S_\nu=a_\nu+b_\nu\tau_\nu$ we find

$$I(0;\mu,\nu)=a_\nu+b_\nu\mu=S(\tau_\nu/\mu=1),\tag{79.17}$$

which is known as the *Eddington-Barbier relation.* This important result shows that the emergent intensity along a ray is approximately equal to the source function at slant optical depth unity along that ray, that is, at about one photon mean free path from the surface.

THE SCHWARZSCHILD-MILNE RELATIONS

We can also use (79.14) and (79.15) to obtain concise expressions for the mean intensity and flux within the medium. Thus the mean intensity is

$$\begin{aligned}J_\nu(\tau_\nu)&=\tfrac12\int_{-1}^{1}I_\nu(\tau_\nu,\mu)\,d\mu\\&=\tfrac12\int_0^1 d\mu/\mu\int_{\tau_\nu}^{\infty}d\tau_\nu'S_\nu(\tau_\nu')e^{-(\tau_\nu'-\tau_\nu)/\mu}-\tfrac12\int_{-1}^{0}d\mu/\mu\int_0^{\tau_\nu}d\tau_\nu'S_\nu(\tau_\nu')e^{(\tau_\nu-\tau_\nu')/\mu}.\end{aligned}\tag{79.18}$$

Substituting $w \equiv \pm\mu^{-1}$ in the first and second integrals respectively, and interchanging the order of integration, we find

$$\begin{aligned} J_\nu(\tau_\nu) &= \tfrac{1}{2}\left[\int_{\tau_\nu}^{\infty} S_\nu(\tau'_\nu)E_1(\tau'_\nu-\tau_\nu)\,d\tau'_\nu + \int_0^{\tau_\nu} S_\nu(\tau'_\nu)E_1(\tau_\nu-\tau'_\nu)\,d\tau'_\nu\right] \\ &= \tfrac{1}{2}\int_0^{\infty} S(\tau'_\nu)E_1\,|\tau'_\nu-\tau_\nu|\,d\tau'_\nu, \end{aligned} \tag{79.19}$$

which was first derived by K. Schwarzschild, and bears his name. Here $E_1(x)$ is the *first exponential integral* from the family

$$E_n(x) \equiv \int_1^{\infty} y^{-n}e^{-xy}\,dy = x^{n-1}\int_x^{\infty} y^{-n}e^{-y}\,dy, \tag{79.20}$$

whose mathematical properties are discussed in (**A1,** Chap. 5).

By a similar analysis we derive

$$H_\nu(\tau_\nu) = \tfrac{1}{2}\left[\int_{\tau_\nu}^{\infty} S_\nu(\tau'_\nu)E_2(\tau'_\nu-\tau_\nu)\,d\tau'_\nu - \int_0^{\tau_\nu} S_\nu(\tau'_\nu)E_2(\tau_\nu-\tau'_\nu)\,d\tau'_\nu\right] \tag{79.21}$$

and

$$K_\nu(\tau_\nu) = \tfrac{1}{2}\int_0^{\infty} S_\nu(\tau'_\nu)E_3\,|\tau'_\nu-\tau_\nu|\,d\tau'_\nu; \tag{79.22}$$

these expressions were first obtained by E. A. Milne (**M5**).

Equations (79.19), (79.21), and (79.22) are used frequently in radiative transfer theory, and are often abbreviated to an operator notation:

$$\Lambda_\tau[f(x)] \equiv \tfrac{1}{2}\int_0^{\infty} f(x)E_1\,|\tau-x|\,dx, \tag{79.23}$$

$$\Phi_\tau[f(x)] \equiv 2\left[\int_\tau^{\infty} f(x)E_2(x-\tau)\,dx - \int_0^{\tau} f(x)E_2(\tau-x)\,dx\right], \tag{79.24}$$

and

$$X_\tau[f(x)] \equiv \tfrac{1}{2}\int_0^{\infty} f(x)E_3\,|\tau-x|\,dx. \tag{79.25}$$

The mathematical properties of these operators are discussed in detail in (**K1,** Chap. 2).

The exponential integrals all have the asymptotic behavior $E_n(x) \sim e^{-x}/x$ for $x \gg 1$. Thus from (79.19) and (79.22) one sees that the values of $J(\tau)$ and $K(\tau)$ for $\tau \gg 1$ are effectively fixed by the value of S over a range $\Delta\tau \approx \pm 1$ from the point in question. In contrast, the flux is given by a differencing operator, its value being determined by the difference between the amount of emission from deeper layers and that from shallower layers.

To illustrate, consider a linear source function $S(\tau) = a + b\tau$. From

(79.14), (79.15), and (79.23) to (79.25) we find

$$I(\tau, \mu) = (a + b\tau) + b\mu, \qquad (0 \le \mu \le 1), \tag{79.26}$$

$$I(\tau, \mu) = (a + b\tau) + b\mu - (a + b\mu)e^{\tau/\mu}, \qquad (-1 \le \mu \le 0), \tag{79.27}$$

$$J(\tau) = \Lambda_\tau[S(t)] = a + b\tau + \tfrac{1}{2}[bE_3(\tau) - aE_2(\tau)], \tag{79.28}$$

$$H(\tau) = \tfrac{1}{4}\Phi_\tau[S(t)] = \tfrac{1}{3}b + \tfrac{1}{2}[aE_3(\tau) - bE_4(\tau)], \tag{79.29}$$

and

$$K(\tau) = \tfrac{1}{4}X_\tau[S(t)] = \tfrac{1}{3}(a + b\tau) + \tfrac{1}{2}[bE_5(\tau) - aE_4(\tau)]. \tag{79.30}$$

From (79.26) to (79.30) we see that at great depth $I(\tau, \mu)$ contains an isotropic component equal to $S(\tau)$ and an anisotropic component proportional to the gradient of S. Similarly, for $\tau \gg 1$, $J(\tau) \rightarrow S(\tau)$ and $K(\tau) \rightarrow \frac{1}{3}S(\tau)$ hence $f = K/J \rightarrow \frac{1}{3}$, as one expects because for $\tau \gg 1$, $I(\tau, \mu)$ becomes essentially isotropic. Furthermore, $H(\tau) \rightarrow \frac{1}{3}b$, which shows explicitly that the flux depends only on the local gradient.

At the surface, boundary effects become important. Noting that $E_n(0) = (n-1)^{-1}$, one finds $J(0) = \frac{1}{2}a + \frac{1}{4}b$, showing that $J(0) < S(0)$ if b is small. In particular for $b = 0$ (isothermal medium), $J(0) = \frac{1}{2}S(0)$ as expected physically because $J(0)$ is then the average of a hemisphere having no radiation ($\mu \le 0$) with one in which $I \equiv S = \text{constant}$. If $b \ge 2a$, then $J(0) \ge S(0)$, and the contribution of photons from deeper, brighter layers outweighs the dilution effects of the hemisphere with no radiation. Similarly $H(0) = \frac{1}{4}a + \frac{1}{6}b$, showing that the surface flux is larger, the faster S increases inward.

Finally, note again that when S contains a scattering term [e.g., equation (77.7)], (79.19) is an integral equation for J.

TIME-DEPENDENT RADIATION FIELD

Suppose now that the material properties and the radiation field are explicitly time dependent. The transfer equation in Cartesian coordinates is then

$$c^{-1}(\partial I/\partial t) + \mathbf{n} \cdot \boldsymbol{\nabla} I = \eta - \chi I. \tag{79.31}$$

Again, the radiation moving in direction $\mathbf{n}$ at $(\mathbf{x}, t)$ consists of photons emitted in direction $\mathbf{n}$ from points along the line $\mathbf{x}' = \mathbf{x} - \mathbf{x}s$ at *retarded times* $t' = t - cs$, where s measures path length backward along the ray. In terms of s, (79.31) can be written

$$\begin{aligned} -\frac{d}{ds}\left[I\left(\mathbf{x} - \mathbf{n}s, t - \frac{s}{c}; \mathbf{n}, \nu\right)\right] + \chi\left(\mathbf{x} - \mathbf{n}s, t - \frac{s}{c}; \mathbf{n}, \nu\right) I\left(\mathbf{x} - \mathbf{n}s, t - \frac{s}{c}; \mathbf{n}, \nu\right) \\ = \eta\left(\mathbf{x} - \mathbf{n}s, t - \frac{s}{c}; \mathbf{n}, \nu\right). \end{aligned} \tag{79.32}$$

Equation (79.32) admits an integrating factor, yielding the indefinite integral

$$I(\mathbf{x}, t; \mathbf{n}, \nu) = \int \eta(\mathbf{x}-\mathbf{n}s', t-s'/c; \mathbf{n}', \nu) \times \exp\left[-\int_0^{s'} \chi(\mathbf{x}-\mathbf{n}s'', t-s''/c; \mathbf{n}, \nu)\, ds''\right] ds'. \tag{79.33}$$

The physical interpretation of (79.33) is straightforward: the intensity at $(\mathbf{x}, t)$ is the integrated contribution of photons emitted in direction $\mathbf{n}$ from all $\mathbf{x}' = \mathbf{x} - \mathbf{n}s$, at retarded times $t' = t - s/c$, attenuated by the time-dependent opacity the photons encounter as they travel from $\mathbf{x}'$ to $\mathbf{x}$.

The solution (79.33) is incomplete because it does not match initial conditions imposed on I at $t=0$, or account for photons entering the boundary surface S. To account for boundary conditions we restrict the range of integration along $\mathbf{n}$ from $s=0$ to $s = s_0 = |\mathbf{x}_S - \mathbf{x}|$, and add a source term localized to s_0 for all $t>0$. To match the initial condition we add a source term at $t=0$ whose value equals $I(t=0)$ for all $s \le s_0$, and restrict the range of integration over the emissivity to retarded times $t' \ge 0$ along the ray. Thus in (79.33) we formally replace η by

$$\tilde{\eta}(s) = \eta(\mathbf{x}-\mathbf{n}s, t-s/c; \mathbf{n}, \nu)H(t-s/c) + (1/c)g(\mathbf{x}; \mathbf{n}, \nu)H(s_0-s)\,\delta(t) + f(\mathbf{x}_S, t; \mathbf{n}, \nu)\,\delta(s-s_0)H(t), \tag{79.34}$$

where H is the Heaviside function [$H(x)=0$, $x \le 0$, and $H(x)=1$, $x>0$], δ is the Dirac function, and f and g are given boundary conditions as in (79.1) and (79.5).

Subsituting (79.34) into (79.33) we obtain

$$\begin{aligned} I(\mathbf{x}, t; \mathbf{n}, \nu) = &\int_0^{s_0} \eta(\mathbf{x}-\mathbf{n}s', t-s'/c; \mathbf{n}, \nu)H(t-s'/c) \\ &\times \exp\left[-\int_0^{s'} \chi(\mathbf{x}-\mathbf{n}s'', t-s''/c; \mathbf{n}, \nu)\, ds''\right] ds' \\ &+ g(\mathbf{x}-\mathbf{n}ct; \mathbf{n}, \nu)\exp\left[-\int_0^{ct} \chi(\mathbf{x}-\mathbf{n}s'', t-s''/c; \mathbf{n}, \nu)\, ds''\right] H(s_0-ct) \\ &+ f(\mathbf{x}-\mathbf{n}s_0, t-s_0/c; \mathbf{n}, \nu)\exp\left[-\int_0^{s_0} \chi(\mathbf{x}-\mathbf{n}s'', t-s''/c; \mathbf{n}, \nu)\, ds''\right] H(ct-s_0). \end{aligned} \tag{79.35}$$

The first term in (79.35) accounts for all photons emitted by the material, suitably attenuated, from retarded times $t'>0$. The second term matches the solution onto the initial conditions at $t'=0$ unless t is so large that the ray already penetrated the boundary surface at a retarded time $t_0' = t - s_0/c > 0$; in this event the third term comes into play and imposes the known boundary condition at $\mathbf{x} = \mathbf{x}_S$ for the correct retarded time t_0'.

80. The Diffusion Limit

In the interior of a star we reach the *diffusion limit* where the optical depth is large, photon mean free paths are small, and photons diffuse through the material in a random walk. Because the radiation is efficiently trapped, and the average temperature gradient in a star is so small, the radiation field is thermal to a high degree of approximation.

Thus for $\tau_\nu \gg 1$, $S_\nu \rightarrow B_\nu$, and in the neighborhood of any chosen τ_ν, S_ν can be represented by

$$S_\nu(t_\nu) = \sum_{n=0}^{\infty} B_\nu^{(n)}(\tau_\nu)(t_\nu - \tau_\nu)^n/n! \tag{80.1}$$

where $B_\nu^{(n)} \equiv (\partial^n B_\nu/\partial\tau_\nu^n)$. Here we tacitly assume the medium is planar because photon mean free paths are so small that curvature effects are negligible. If we substitute (80.1) into (79.14), for $0 \leq \mu \leq 1$ we find

$$I_\nu(\tau_\mu, \mu) = B_\nu(\tau_\nu) + \mu(\partial B_\nu/\partial\tau_\nu) + \mu^2(\partial^2 B_\nu/\partial\tau_\nu^2) + \dots. \tag{80.2}$$

The result for $-1 \leq \mu \leq 0$ differs from (80.2) only by terms of $O(e^{-\tau_\nu/|\mu|})$, hence for $\tau_\nu \gg 1$ we can use (80.2) for the full range $-1 \leq \mu \leq 1$. From (64.2), (65.5), and (66.8) we then find

$$J_\nu(\tau_\nu) = B_\nu(\tau_\nu) + \tfrac{1}{3}(\partial^2 B_\nu/\partial\tau_\nu^2) + \dots \tag{80.3}$$

$$H_\nu(\tau_\nu) = \tfrac{1}{3}(\partial B_\nu/\partial\tau_\nu) + \tfrac{1}{5}(\partial^3 B_\nu/\partial\tau_\nu^3) + \dots \tag{80.4}$$

and

$$K_\nu(\tau_\nu) = \tfrac{1}{3}B_\nu(\tau_\nu) + \tfrac{1}{5}(\partial^2 B_\nu/\partial\tau_\nu^2) + \dots. \tag{80.5}$$

We now ask how quickly these series converge. To obtain order-of-magnitude estimates of the derivatives we approximate them by difference quotients: $(\partial^n B_\nu/\partial\tau_\nu^n) \sim B_\nu/\tau_\nu^n$. Then the ratio of successive terms in (80.3) to (80.5) is $O(1/\tau_\nu^2) = O(\lambda_\nu^2/l^2)$ where l is a characteristic structural length. In a stellar envelope a conservative estimate of l is H, the pressure scale height, which in the Sun ranges from 10^2 km near the surface to $\gtrsim 10^3$ km in the interior. Thus representative values for (λ_ν/l) lie on the range 10^{-7} to 10^{-10}, implying a convergence factor of order 10^{-14} to 10^{-20} for the series. Hence we need only the first terms of (80.3) to (80.5), but we must retain two terms in (80.2) because it is the small asymmetry produced by the gradient that yields a nonzero flux.

Thus, well inside an opaque, static medium an asymptotic solution of the transfer equation is

$$J_\nu(\tau_\nu) = 3K_\nu(\tau_\nu) = B_\nu(\tau_\nu) \tag{80.6}$$

and

$$H_\nu = \tfrac{1}{3}(\partial B_\nu/\partial\tau_\nu) = -\tfrac{1}{3}(\partial B_\nu/\partial T)(dT/dr)/\chi_\nu. \tag{80.7}$$

These results are consistent with (79.28) to (79.30) for $\tau \gg 1$; (80.7) also follows immediately from (78.19) for $f_\nu \equiv \frac{1}{3}$. Thus far we have shown only that (80.6) and (80.7) apply in a static medium; we will see in §97 that they also apply in the comoving frame of a moving medium when velocity-gradient terms are neglected.

Equations (80.6) show that in the diffusion limit the radiation energy density and radiation pressure have their equilibrium values despite the anisotropy in the radiation field; only if we carry terms of $O(\lambda_p^2/l^2)$ in (80.3) and (80.5) is there a departure from equilibrium, but, as we have seen, such terms are truly negligible. Furthermore, (80.7) shows that the radiation flux is also a local quantity that depends on the local temperature gradient. Indeed, integrating (80.7) over frequency we find

$$F = (L/4\pi r^2) = -(4\pi/3)\left[\int_0^\infty \chi_\nu^{-1}(\partial B_\nu/\partial T)\,d\nu\right](dT/dr), \qquad (80.8)$$

which shows that for $\tau_\nu \gg 1$ the radiant heat flux has precisely the same mathematical form, $\mathbf{F} = -K_R \nabla T$, as molecular heat conduction in a gas. In terms of the *Rosseland mean opacity* χ_R (cf. §82) defined by

$$\chi_R^{-1} \equiv \left[\int_0^\infty \chi_\nu^{-1}(\partial B_\nu/\partial T)\,d\nu\right] \Big/ \int_0^\infty (\partial B_\nu/\partial T)\,d\nu, \qquad (80.9)$$

the effective *radiative conductivity* is

$$K_R = (4\pi/3\chi_R)(dB/dT) = \tfrac{4}{3}c\lambda_R a_R T^3, \qquad (80.10)$$

where λ_R is the mean free path corresponding to χ_R. This expression for K_R is exactly what one expects from the mean-free-path arguments of §29: the conductivity is proportional to the product of (1) the energy density associated with the transporting particles (photons) divided by the temperature, (2) the particle speed, and (3) the particle mean free path [cf. (29.23)].

Insofar as it predicts that the energy density and hydrostatic pressure of the radiation field are fixed by the local temperature, while the radiative flux is proportional to the temperature gradient, the diffusion-limit solution of the transfer equation is closely analogous to the first-order Chapman-Enskog solution of the Boltzmann equation for material gases. For moving media the analogy can be pushed even further (cf. §97), and one finds that P contains viscous terms proportional to the rate of strain tensor.

It is important to note that the diffusion-limit flux is a very small leak compared to the radiation energy density. Thus define the *effective temperature* $T_{\rm eff}$ of a star such that

$$L = 4\pi R^2 \sigma_R T_{\rm eff}^4 \qquad (80.11)$$

where L and R are the stellar luminosity and radius; from (68.9), $T_{\rm eff}$ is the temperature of an equivalent black body of radius R that radiates a total luminosity L. The emergent flux is

$$F = \sigma_R T_{\rm eff}^4; \qquad (80.12)$$

this net energy flux ($\mathrm{erg\,cm^{-2}\,s^{-1}}$) is transported by particles (photons) that have velocity c, hence the effective energy density associated with the flux is F/c. The total radiation energy density in the field is $E=(4\pi/c)B=4\sigma_R T^4/c$, hence

$$\frac{\text{(Effective energy density in radiation flow)}}{\text{(Total energy density of radiation)}} \sim \frac{(\sigma_R T_{\text{eff}}^4/c)}{(\sigma_R T^4/c)} = \left(\frac{T_{\text{eff}}}{T}\right)^4. \tag{80.13}$$

For a star like the Sun, $T_{\text{eff}} \approx 6\times 10^3$ K, whereas interior temperatures are $\sim 10^7$ K. Thus in the solar interior the leak is about one part in 10^{12}; very small indeed! In contrast, near the surface $T \approx T_{\text{eff}}$, and the energy density associated with freely escaping photons is the energy density of the field itself.

From (80.8) and (80.9) we obtain one of the standard equations of stellar structure, that is,

$$\frac{dT}{dM_r} = \frac{-3\chi_R}{16\sigma_R \rho T^3}\frac{L_r}{(4\pi r^2)^2}, \tag{80.14}$$

where $dM_r = 4\pi r^2 \rho\, dr$ is the mass in a shell of radius r and thickness dr, and L_r is the luminosity passing through a sphere of radius r. In stellar interiors work (80.14) is viewed as an equation that determines the temperature gradient; but it has deeper significance as an asymptotic solution of the radiative transfer equation. Notice that dM_r is a material element, hence M_r can be used as a Lagrangean variable if the material moves (e.g., stellar pulsation). We show in §97 that (80.14) remains valid in moving material provided that we neglect terms of $O(\lambda_p v/lc)$, and we measure *all* quantities, including L_r (or F_r), in the comoving fluid frame.

Similarly the time-independent energy equation (78.8) for a static medium can be rewritten as

$$(dL_r/dM_r) = (4\pi/\rho)\int_0^\infty (\eta_\nu - \chi_\nu J_\nu)\,d\nu = (4\pi/\rho)\int_0^\infty \chi_\nu (S_\nu - J_\nu)\,d\nu. \tag{80.15}$$

If we adopt a source function of the form (77.8), scattering terms cancel identically and we obtain

$$(dL_r/dM_r) = (4\pi/\rho)\int_0^\infty \kappa_\nu (B_\nu - J_\nu)\,d\nu, \tag{80.16}$$

where κ_ν is the thermal absorption coefficient, and in writing η_ν we assumed LTE. One might expect from (80.6) that in the diffusion limit the right-hand side of (80.16) will vanish identically. Physically this would correspond to a state of radiative equilibrium (cf. §78), in which the material emits exactly the amount of energy it absorbs, and the luminosity is independent of radius. However, in the deep interior there can also be an irreversible release of thermonuclear energy at a rate of ε ergs $\text{gm}^{-1}\,\text{s}^{-1}$, which drives a *luminosity gradient.* The thermonuclear energy released in a mass shell is $\varepsilon\, dM_r$, which for a steady state must equal the change in luminosity across that shell; accounting for this additional term (80.16) reduces to its standard stellar interiors form

$$(dL_r/dM_r) = \varepsilon. \tag{80.17}$$

A more revealing derivation of (80.17) in §96 shows that it is only the static limit of a more general energy equation representing the first law of thermodynamics for the composite matter-radiation gas [see (96.10) and (96.11)].

Finally, we show that in the limit of large optical depth, the transfer equation can be manipulated into a time-dependent diffusion equation for the energy density. First, in the diffusion regime we can neglect the term $c^{-2}(\partial \mathbf{F}_\nu/\partial t)$ in (78.10) compared to the right-hand side which, in the fluid frame where χ and η are isotropic, reduces to $(\chi_\nu \mathbf{F}_\nu/c)$, because the time required for photons to random walk a distance l is $\Delta t \sim (l/\lambda_p)^2(\lambda_p/c) = l^2/\lambda_p c$, hence the ratio of the terms in question is

$$c^{-2}(\partial F_\nu/\partial t)/(\chi_\nu F_\nu/c) \sim (1/\chi c\, \Delta t) = (\lambda_p/l)^2 \sim 10^{-18} \tag{80.18}$$

for typical values $l \sim H = 10^3$ km and $\lambda_p \sim 10^{-1}$ cm. Thus we can use the static form of (78.11) restated as

$$\mathbf{F}_\nu = -c\chi_\nu^{-1}\,\boldsymbol{\nabla} P_\nu = -\tfrac{1}{3}c\chi_\nu^{-1}\,\boldsymbol{\nabla} E_\nu, \tag{80.19}$$

where we made use of the isotropy of P_ν. Substituting (80.19) into (78.3) we have (for a static medium)

$$\frac{1}{c}\frac{\partial E_\nu}{\partial t} = \frac{1}{3}\boldsymbol{\nabla}\cdot\left(\frac{1}{\chi_\nu}\boldsymbol{\nabla} E_\nu\right) + \left(\frac{4\pi}{c}\right)\eta_\nu - \chi_\nu E_\nu, \tag{80.20}$$

which, as asserted, has the same form as the time-dependent diffusion equation $(\partial f/\partial t) = \nabla^2 f + \mathcal{S}$ where $\mathcal{S}$ is a source-sink term.

81. The Wave Limit

Having seen that, at large optical depth, the transfer equation behaves like a diffusion equation, we now show that in a vacuum ($\chi_\nu = \eta_\nu = 0$) it reduces to the wave equation, which is to be expected from classical electrodynamics. Although one obtains a perfectly unattenuated wave only in true vacuum, the radiation field also approaches this free streaming limit in optically thin media.

In the absence of material the transfer equation (76.3) becomes

$$(\partial I/\partial t) + c(\partial I/\partial s) = 0 \tag{81.1}$$

where s measures the pathlength along the ray $\mathbf{n}$. Defining $I^+ \equiv I(\mathbf{x}, t; \mathbf{n}, \nu)$

and $I^- \equiv I(\mathbf{x}, t; -\mathbf{n}, \nu)$, (81.1) yields

$$(\partial I^+/\partial t) + c(\partial I^+/\partial s) = 0 \tag{81.2}$$

and

$$(\partial I^-/\partial t) - c(\partial I^-/\partial s) = 0. \tag{81.3}$$

Now define the mean-intensity-like quantity

$$j \equiv \tfrac{1}{2}(I^+ + I^-) \tag{81.4}$$

and the fluxlike quantity

$$h \equiv \tfrac{1}{2}(I^+ - I^-); \tag{81.5}$$

then (81.2) and (81.3) can be added and subtracted to produce

$$(\partial j/\partial t) + c(\partial h/\partial s) = 0 \tag{81.6}$$

and

$$(\partial h/\partial t) + c(\partial j/\partial s) = 0. \tag{81.7}$$

Equations (81.6) and (81.7) combine into the wave equations

$$(\partial^2 j/\partial t^2) = c^2(\partial^2 j/\partial s^2) \tag{81.8}$$

and

$$(\partial^2 h/\partial t^2) = c^2(\partial^2 h/\partial s^2), \tag{81.9}$$

which have the solutions

$$j(s, t) = A_1 f_1(s - ct) + A_2 f_2(s + ct) \tag{81.10}$$

and

$$h(s, t) = B_1 f_1(s - ct) + B_2 f_2(s + ct). \tag{81.11}$$

We recognize these as traveling waves moving along $\pm\mathbf{n}$. As usual, the constants $A_1, \ldots, B_2$ are determined by initial and boundary conditions.

Equations (81.10) and (81.11) imply that one can construct a particular solution of the form

$$I(\mathbf{x}, t; \mathbf{n}', \nu') = I_0\, \delta(s - ct)\, \delta(\mathbf{n}' - \mathbf{n})\, \delta(\nu' - \nu), \tag{81.12}$$

that is, a monochromatic plane wave traveling along $\mathbf{n}$ with velocity c. As noted in §66, for such a wave $J_\nu = H_\nu = K_\nu$.

Wave equations also follow from the moment equations once we know that solutions of the form (81.12) exist. Thus if we choose $\mathbf{n} = \mathbf{k}$ in planar geometry, (78.3) and (78.10) become

$$(\partial J_\nu/\partial t) + c(\partial H_\nu/\partial z) = 0 \tag{81.13}$$

and

$$(\partial H_\nu/\partial t) + c(\partial K_\nu/\partial z) = 0, \tag{81.14}$$

and because $J_\nu = K_\nu$ for a plane wave, (81.13) and (81.14) combine to give

$$(\partial^2 J_\nu/\partial t^2) + c^2(\partial^2 J_\nu/\partial z^2) = 0 \tag{81.15}$$

and

$$(\partial^2 H_\nu/\partial t^2)+c^2(\partial^2 H_\nu/\partial z^2)=0, \tag{81.16}$$

which are standard wave equations in planar geometry.

In spherical geometry (78.5) becomes

$$(\partial J_\nu/\partial t)+(c/r^2)[\partial(r^2 H_\nu)/\partial r]=0, \tag{81.17}$$

while (78.11) reduces to

$$(\partial H_\nu/\partial t)+(c/r^2)[\partial(r^2 J_\nu)/\partial r]=0 \tag{81.18}$$

when we recall (66.10) and demand $K_\nu = J_\nu$. Equations (81.17) and (81.18) combine to yield

$$(\partial^2 J_\nu/\partial t^2)=(c^2/r^2)[\partial^2(r^2 J_\nu)/\partial r^2]=c^2\nabla^2 J_\nu \tag{81.19}$$

and

$$(\partial^2 H_\nu/\partial t^2)=c^2\nabla^2 H_\nu, \tag{81.20}$$

which are standard wave equations in spherical geometry.

82. The Grey Atmosphere, Mean Opacities, and Multigroup Methods

MOTIVATION AND ASSUMPTIONS

We now consider a highly simplified problem, which provides valuable experience in solving the transfer equation: radiative transfer and energy balance in a static LTE medium composed of *grey material* (one whose absorption coefficient is independent of frequency). This problem can be solved relatively easily and completely, and yields reasonable estimates of the run of the physical properties in the outer layers of stars, thus giving (1) moderately accurate boundary conditions for stellar envelope and interior calculations, and (2) starting solutions for iterative methods for handling more accurate treatments of the physics.

For grey material $\chi_\nu \equiv \chi$, and the transfer equation (77.5) becomes

$$\mu(\partial I_\nu/\partial\tau)=I_\nu-S_\nu. \tag{82.1}$$

Integrating over frequency we have

$$\mu(\partial I/\partial\tau)=I-S, \tag{82.2}$$

where quantities such as I, J, H, K, B, and S without subscripts denote frequency-integrated variables, for example,

$$I\equiv\int_0^\infty I_\nu\,d\nu. \tag{82.3}$$

Because the medium is static, it must be in radiative equilibrium; hence

$$4\pi\int_0^\infty \chi_\nu J_\nu\,d\nu=4\pi\int_0^\infty \chi_\nu S_\nu\,d\nu, \tag{82.4}$$

which, for grey material, reduces to $J \equiv S$. Thus the transfer equation to be solved is

$$\mu(\partial I/\partial\tau) = I - J, \tag{82.5}$$

which is a *homogeneous* integrodifferential equation for I, posing what is called *Milne's problem*. The Milne problem is to be solved for $J(\tau)$ [hence $S(\tau)$] from which we compute $I(\tau, \mu)$ and $K(\tau)$. We already know that $H(\tau)$ is constant in radiative equilibrium [cf. (82.7)].

With the additional assumption of LTE, $S_\nu \equiv B_\nu$, hence

$$B[T(\tau)] = \sigma_R T^4/\pi = S(\tau) = J(\tau). \tag{82.6}$$

Therefore if we can determine $J(\tau)$ from (82.5), then (82.6) allows us to associate a temperature with the radiation field at each depth.

GREY MOMENT EQUATIONS

Calculating the zeroth moment of (82.5) we obtain

$$(dH/d\tau) = J - S = J - J \equiv 0, \tag{82.7}$$

which shows that the flux is indeed constant. The first moment yields

$$(dK/d\tau) = H, \tag{82.8}$$

which has the exact integral

$$K(\tau) = H(\tau + C), \tag{82.9}$$

where C is a constant. For $\tau \gg 1$, $J(\tau) \to 3K(\tau)$; hence (82.9) implies that when $\tau \gg 1$, $J(\tau) \approx 3H\tau$. This result suggests a general expression for $J(\tau)$ of the form

$$J(\tau) = 3H[\tau + q(\tau)], \tag{82.10}$$

where $q(\tau)$, the *Hopf function*, is a bounded function, to be determined. In terms of $q(\tau)$, (82.9) becomes

$$K(\tau) = H[\tau + q(\infty)]. \tag{82.11}$$

The solution of the grey problem consists of the calculation of $q(\tau)$. Given this function we can determine the run of temperature with depth from

$$T^4 = \tfrac{3}{4} T_{\text{eff}}^4 [\tau + q(\tau)], \tag{82.12}$$

which follows from (82.6), (82.10), and (80.12).

A wide variety of methods have been developed for determining $q(\tau)$; these are discussed at length in (**C6**) and (**K1**); in fact, it is possible to obtain an exact solution in closed form. We shall not review this large literature here, but will discuss only two methods: the simplest approximation, which yields roughly the right answer, and a second, which yields accurate answers for the grey problem and also provides the basic approach used in solving more complex transfer problems.

THE EDDINGTON APPROXIMATION

To obtain an approximate solution of the grey problem, Eddington made the simplifying assumption that the relation $J(\tau) = 3K(\tau)$, valid at great depth, holds throughout the entire medium. Although Eddington's approximation is not exact, it is nevertheless reasonable. For example, the relation $J = 3K$ holds (1) in the diffusion regime where $I(\tau, \mu) = I_0(\tau) + I_1(\tau)\mu$; and (2) in the *two-stream approximation* where $I(\tau, \mu) \equiv I^+(\tau)$ for $0 \leq \mu \leq 1$ and $I(\tau, \mu) \equiv I^-(\tau)$ for $-1 \leq \mu \leq 0$ for arbitrary (but μ-independent) values of I^+ and I^-. The latter provides a rough representation of the radiation field near the boundary of a semi-infinite medium, for we may let $I^-/I^+ \to 0$ as $\tau \to 0$, and $I^-/I^+ \to 1$ for $\tau \gtrsim 1$.

From (82.10) and (82.11) we see that Eddington's approximation is equivalent to writing

$$J_E(\tau) = 3H(\tau + C'), \tag{82.13}$$

where the constant C' is to be determined. To fix C' we use $S = J_E$ in (79.21) to calculate the emergent flux, obtaining

$$H(0) = \tfrac{3}{2}H \int_0^\infty (\tau + C')E_2(\tau)\, d\tau = \tfrac{3}{2}H[E_4(0) + C'E_3(0)] = \tfrac{3}{2}H(\tfrac{1}{3} + \tfrac{1}{2}C'). \tag{82.14}$$

Demanding $H(0) \equiv H$ we find $C' = \frac{2}{3}$. Thus in the Eddington approximation $q_{\mathrm{E}}(\tau) \equiv \frac{2}{3}$,

$$J_{\mathrm{E}}(\tau) = 3H(\tau + \tfrac{2}{3}) \tag{82.15}$$

and

$$T^4 = \tfrac{3}{4}T_{\mathrm{eff}}^4(\tau + \tfrac{2}{3}). \tag{82.16}$$

The exact solution of the grey problem gives $q(0) = 1/\sqrt{3} = 0.577\ldots$, and $q(\infty) = 0.710\ldots$, compared with $q_{\mathrm{E}} \equiv 0.666\ldots$. Not surprisingly, the error in q_E and J_E is largest at the surface; one finds $\Delta J(0)/J_{\mathrm{exact}}(0) \approx 0.15$. On the other hand, (82.16) predicts $(T_0/T_{\mathrm{eff}})_{\mathrm{E}} = (\frac{1}{2})^{1/4} \approx 0.841$, which agrees well with $(T_0/T_{\mathrm{eff}})_{\mathrm{exact}} = (\frac{1}{4}\sqrt{3})^{1/4} \approx 0.811$. Thus Eddington's approximation does provide a good first estimate for the temperature structure of a grey atmosphere. Note also that (82.16) predicts $T = T_{\mathrm{eff}}$ at $\tau = \frac{2}{3}$; for this reason $\tau = \frac{2}{3}$ is often considered to be the effective depth of continuum formation in a semi-infinite medium.

Furthermore, using (82.15) in (79.16) we find

$$I_{\mathrm{E}}(\tau = 0, \mu) = H(2 + 3\mu), \tag{82.17}$$

which shows that the radiation field is peaked in the direction of outward flow; indeed $I_{\mathrm{E}}(\mu = 1)/I_{\mathrm{E}}(\mu = 0) = 2.5$. Using (82.17) to calculate J and K at $\tau = 0$ we find $J_{\mathrm{E}}(0) = \frac{7}{4}H$ and $K_E(0) = \frac{17}{24}H$, whence the variable Eddington factor at the surface is $f_{\mathrm{E}}(0) = \frac{17}{42} \approx 0.405$; the exact solution yields $f(0) = q(\infty)/3q(0) \approx 0.410$. These results are important because they show that using even a rough estimate ($f \equiv \frac{1}{3}$) of the Eddington factor to solve the transfer equation we obtain a source function that yields a reasonably accurate angular distribution of the intensity, from which we can compute a much better estimate of f throughout the entire atmosphere. We return to this point in §83.

THE METHOD OF DISCRETE ORDINATES

While Eddington's approach gives useful results, it lacks accuracy and generality. We can also solve the grey problem by rewriting (82.6) explicitly as an integrodifferential equation

$$\mu[\partial I(\tau,\mu)/\partial\tau]=I(\tau,\mu)-\tfrac{1}{2}\int_{-1}^{1} I(\tau,\mu)\,d\mu, \qquad (82.18)$$

and approximating the integral as a *quadrature sum.* In this procedure a function $f(\mu)$ defined on $-1\le\mu\le1$ is sampled at a set of *quadrature points* $\{\mu_m\}$, $(m=\pm1,\ldots,\pm M)$, where $0\le\mu_m\le1$ and $\mu_{-m}=-\mu_m$. Applying standard techniques of numerical analysis, one can generate a set of *quadrature weights* $\{b_m\}$ defined such that the definite integral in (82.18) is represented as a weighted sum over the *discrete ordinates* $\{f(\mu_m)\}$, that is,

$$\int_{-1}^{1} f(\mu)\,d\mu\approx\sum_{m=-M}^{M} b_m f(\mu_m). \qquad (82.19)$$

This procedure assumes, in effect, that $f(\mu)$ is the unique interpolating polynomial of order $2M-1$ that passes through the $2M$ ordinates $\{f(\mu_m)\}$.

For the transfer problem we thus represent the angular variation of $I(\tau,\mu)$ by a set of *pencils* of radiation $I_i\equiv I(\tau,\mu_i)$, and replace the integrodifferential equation (82.18) by a coupled set of ordinary differential equations:

$$\mu_i(dI_i/d\tau)=I_i-\tfrac{1}{2}\sum_{m=-M}^{M} b_m I_m, \qquad (m=\pm1,\ldots,\pm M). \qquad (82.20)$$

One thinks of each pencil I_m as giving an average of $I(\mu)$ over a definite range of μ around μ_m. On physical grounds it is reasonable to expect the solution to become increasingly accurate as M increases, and to limit the exact solution as $M\to\infty$.

The *method of discrete ordinates* described above provides an extremely powerful tool for solving transfer problems, and it will be exploited heavily in §§83 and 88, and in Chapter 7. For the grey problem, Chandrasekhar (**C6**) obtained a complete analytical solution of (82.20), which matches the boundary conditions and gives constant flux; furthermore, by studying the limit $M\to\infty$ he deduced many properties of the exact $q(\tau)$. The full exact solution was first obtained by completely different (Laplace transform) methods, discussed in (**K1**) and (**M2,** Chap. 3).

The accuracy of a quadrature formula depends both on the number of quadrature points used, and on their distribution within the interval. A good discussion of methods for constructing quadrature formulae is given in (**C6,** Chap. 2). In *Newton-Cotes formulae* the $\{\mu_m\}$ are equally spaced; for $2M$ points the quadrature is exact if $I(\mu)$ is a polynomial of order $\leq 2M-1$. A more favorable choice is the *Gauss formula,* in which the $\{\mu_m\}$ are the roots of the Legendre polynomial P_{2M}, and which is exact if $I(\mu)$ is a polynomial of order $\leq 4M-1$. An even better choice for transfer problems is the *double-Gauss formula* suggested by Sykes (**S5**), which is now universally used. Here one uses a separate M-point Gauss formula on each of the subintervals $[-1, 0]$ and $[0, 1]$, the points being the roots of P_M suitably shifted and scaled from $[-1, 1]$ to each subinterval. (An important exception is that for $M=1$ one must choose $\mu_{\pm 1}=\pm 1/\sqrt{3}$.) The quadrature is exact if $I(\mu)$ is a polynomial of order $\leq 2M-1$ on each subinterval. Although for a given number of points the formal accuracy of the double-Gauss formula is lower on each subinterval than the ordinary Gauss formula, for transfer problems it is vastly superior because $I(+\mu)$ and $I(-\mu)$ are approximated independently, hence it can account for the fact that as $\tau \rightarrow 0$, $I(-\mu) \rightarrow 0$ while $I(+\mu)$ remains finite. The ordinary Gauss formula spanning $[-1, 1]$ tries, in effect, to integrate through the discontinuity at $\mu=0$, and naturally loses accuracy in doing so. Quadrature points and weights for double-Gauss formulae are given in (**A1,** 921).

THE NONGREY PROBLEM

The opacity frequency-spectrum of real material is complicated (cf. §72), and in solving realistic transfer problems we confront the difficult question of how best to model this spectrum. Several approaches have been developed; each has strengths and weaknesses, which, for any particular problem, must be weighed carefully. Typically a compromise must be made between accuracy and economy of computation.

One obvious approach is the *direct method* in which a large number of frequency points are chosen so as to represent all the major features of the opacity (e.g., continuum edges and strong lines); one then solves the transfer equation or the moment equations at each of these frequencies. This approach is satisfactory for material with a relatively simple spectrum (e.g., in hot stars where the dominant processes are bound-free and free-free absorption by H, H^+, He, He^+, and He^{++}; Thomson scattering by free electrons; and absorption in a small number of strong lines) and in such cases it can yield accurate results. But for complex spectra (e.g., in cool stars with millions of atomic and molecular lines) the direct method is prohibitively costly, especially for dynamical problems. Let us therefore consider alternatives.

MEAN OPACITIES

The simplest possible solution of the nongrey problem would be obtained if a single *mean opacity* could represent correctly the total transport of radiation through the material; the nongrey problem would then reduce to an equivalent grey problem, whose solution is known. Not surprisingly, such a reduction is not possible in general, basically because opacities enter *nonlinearly* in transport processes. Nevertheless, certain mean opacities have important physical significance and permit considerable simplification of the nongrey problem in some regimes, while providing useful approximations in others where they are not rigorously correct. Here we discuss only planar geometry, but mean opacities can be used in any geometry.

(*a*) *The Rosseland Mean* The key role usually played by radiation in a radiating fluid is the transport of energy. Can we define a mean opacity that guarantees the correct total transport of radiant energy? We will obtain the correct total flux from the first moment equation (78.16) if $\bar{\chi}$ is chosen such that

$$-\int_0^\infty \frac{1}{\chi_\nu}\frac{\partial K_\nu}{\partial z}\,d\nu = \int_0^\infty H_\nu\,d\nu = H \equiv -\frac{1}{\bar{\chi}}\frac{dK}{dz} = -\frac{1}{\bar{\chi}}\int_0^\infty \frac{\partial K_\nu}{\partial z}\,d\nu. \quad (82.21)$$

or

$$\bar{\chi} \equiv \int_0^\infty (\partial K_\nu/\partial z)\,d\nu \Big/ \int_0^\infty \chi_\nu^{-1}(\partial K_\nu/\partial z)\,d\nu. \quad (82.22)$$

The difficulty in obtaining the correct $\bar{\chi}$ in general is clear from (82.22): we must know K_ν to compute $\bar{\chi}$, but to determine K_ν we must solve the full nongrey problem.

However, in the diffusion regime $K_\nu \to \frac{1}{3}B_\nu$, and $\bar{\chi}$ then reduces to the *Rosseland mean* defined by (80.9). Thus in the diffusion regime we *can* replace an arbitrarily complex opacity spectrum by a single average that guarantees the correct radiative energy transport; this is why the Rosseland mean is universally used in stellar interiors work. Furthermore, in LTE, χ_ν is a function of T and ρ, while B_ν is a function of T only; hence $\chi_{\rm R}$ can be computed once and for all as a function of local state variables.

Note that $\chi_{\rm R}$ is a *harmonic mean*, giving greatest weight to the most transparent regions of the spectrum. Thus opaque features (e.g., strong lines) affect $\chi_{\rm R}$ mainly by reducing the *bandwidth* through which efficient energy transport can occur. A change in the absolute strength of opaque features has little or no effect on $\chi_{\rm R}$; but the addition of a continuum source and/or many faint lines to an otherwise weakly absorbing spectral region raises the minimum value of χ_ν there, and can increase $\chi_{\rm R}$ significantly.

From (82.21) it follows that, even when the material is not grey, in the diffusion limit $(dK/d\tau_R) = \frac{1}{3}(dB/d\tau_R) = H$, where $H = (\sigma_R T_{\rm eff}^4/4\pi)$, and $\tau_R = -\int \chi_R\,dz$. Hence the temperature distribution for $\tau_{\rm R} \gg 1$ in a static nongrey medium in radiative equilibrium is accurately given by the modified grey relation

$$T^4 = \tfrac{3}{4}T_{\rm eff}^4[\tau_{\rm R} + q(\tau_R)]. \quad (82.23)$$

Detailed calculations for nongrey radiative-equilibrium atmospheres show that (82.23) is in fact an excellent approximation at depth, and provides a good starting estimate for iterative methods (cf. §88) at the surface. Of course, (82.23) cannot guarantee flux conservation when the diffusion approximation breaks down; thus near a boundary surface the Rosseland mean may seriously underestimate the effective opacity and may yield poor results for radiative energy balance (**C3**).

(*b*) *The Flux Mean* Instead of energy transport we might focus on momentum balance and choose a mean that gives the correct radiation force on the material. From (78.12) and (78.17) the radiation force on a static medium is

$$f_R = (4\pi/c) \int_0^\infty \chi_\nu H_\nu \, d\nu; \tag{82.24}$$

the same expression holds for moving material if all quantities are measured in the comoving frame (cf. §96). Thus the correct radiation force results if $\bar{\chi}$ is defined by

$$\int_0^\infty \chi_\nu H_\nu \, d\nu = \bar{\chi} \int_0^\infty H_\nu \, d\nu = \bar{\chi} H, \tag{82.25}$$

whence

$$\chi_H \equiv \int_0^\infty \chi_\nu H_\nu \, d\nu / H, \tag{82.26}$$

which is called the *flux mean.*

Like the Rosseland mean, χ_H does not allow a complete reduction of the nongrey problem to an equivalent grey problem. Moreover we cannot compute χ_H until we know H_ν, which is obtained only by solving the full nongrey problem. But in the diffusion regime (80.7) implies that χ_H is identical to the Rosseland mean:

$$(\chi_H)_{\text{diffusion}} = \int_0^\infty (\partial B_\nu/\partial T) \, d\nu \Big/ \int_0^\infty \chi_\nu^{-1} (\partial B_\nu/\partial T) \, d\nu \equiv \chi_R. \tag{82.27}$$

Thus in the diffusion limit, the Rosseland mean yields not only the correct energy transport, but the correct momentum balance as well. For this reason the Rosseland mean is often chosen as a representative opacity for use in the momentum equations in problems of radiation hydrodynamics.

(*c*) *The Planck Mean and The Absorption Mean* Other definitions of mean opacities result from requiring correct values for the total energy emitted or absorbed by the material. For a static LTE medium the right-hand side of the radiation energy equation (78.4) reduces to $4\pi \int_0^\infty \kappa_\nu (B_\nu - J_\nu) \, d\nu$ even when scattering terms are present. The same expressions holds for moving media provided that all quantities are measured in the comoving frame (cf. §96). To obtain the correct total emission we thus define a mean opacity $\bar{\kappa}$ such that

$$\int_0^\infty \kappa_\nu B_\nu \, d\nu = \bar{\kappa} \int_0^\infty B_\nu \, d\nu \equiv \kappa_P B(T) \tag{82.28}$$

or

$$\kappa_P \equiv \int_0^\infty \kappa_\nu B_\nu \, d\nu / (\sigma_R T^4/\pi), \tag{82.29}$$

which is called the *Planck mean*. Notice that, like the Rosseland mean, κ_P can be computed once and for all as a function of ρ and T.

To obtain the correct total absorption we must use the *absorption mean* κ_J defined by

$$\kappa_J \equiv \int_0^\infty \kappa_\nu J_\nu \, d\nu \bigg/ \int_0^\infty J_\nu \, d\nu. \tag{82.30}$$

But, like χ_H, κ_J cannot be evaluated unless we have solved the full nongrey transfer problem. It is therefore important that we can show that, in the optically thin regime, κ_P provides a reasonable estimate of the total absorption and thus serves as a useful substitute for κ_J, just as χ_R does for χ_H in the diffusion regime.

In particular, to achieve radiative equilibrium we should choose $\bar{\kappa}$ such that

$$\int_0^\infty \kappa_\nu (B_\nu - J_\nu) \, d\nu = 0 \equiv \bar{\kappa} \int_0^\infty (B_\nu - J_\nu) \, d\nu. \tag{82.31}$$

When the material is transparent ($\tau_\nu \ll 1$ at all frequencies), J_ν is essentially fixed, and the integrals in (82.31) are dominated by the frequencies at which $\kappa_\nu \gg \bar{\kappa}$. For $\bar{\tau} \lesssim 1$ we can represent B_ν by a linear expansion

$$B_\nu(t) = B_\nu(\bar{\tau}) + (\partial B_\nu/\partial \bar{\tau})(t - \bar{\tau}) \approx B_\nu(\bar{\tau}) + (\bar{\kappa}/\kappa_\nu)(\partial B_\nu/\partial \bar{\tau})(t_\nu - \tau_\nu), \tag{82.32}$$

whence, by application of the Λ operator, we obtain

$$J_\nu(\bar{\tau}) \approx B_\nu(\bar{\tau})[1 - \tfrac{1}{2}E_2(\bar{\tau})] + \tfrac{1}{2}(\bar{\kappa}/\kappa_\nu)(\partial B_\nu/\partial \bar{\tau})[E_3(\tau_\nu) + \tau_\nu E_2(\tau_\nu)]. \tag{82.33}$$

In the limit $\bar{\tau} \to 0$, $E_2 \to 1$ and $E_3 \to \frac{1}{2}$, and $(B_\nu - J_\nu) \approx \frac{1}{2}B_\nu - \frac{1}{4}(\bar{\kappa}/\kappa_\nu)(\partial B_\nu/\partial \bar{\tau})$. The second term is least important when $\kappa_\nu \gg \bar{\kappa}$, that is, precisely when the first term makes the largest contribution in (82.31). Hence we most nearly achieve energy balance in optically thin material if $\bar{\kappa}$ satisfies

$$\tfrac{1}{2}\bar{\kappa} \int_0^\infty B_\nu \, d\nu = \tfrac{1}{2} \int_0^\infty \kappa_\nu B_\nu \, d\nu, \tag{82.34}$$

that is, if $\bar{\kappa} = \kappa_P$.

Thus κ_P is a good representative opacity for use in the radiation energy equation. On the other hand, use of κ_P in the first moment equation does not yield the correct flux in the diffusion limit, and we again conclude that no one mean opacity completely reduces the nongrey problem to a grey problem.

MEAN-OPACITY REPRESENTATION OF THE MOMENT EQUATIONS

In LTE,

$$(dH/dz) = \kappa_P B - \kappa_J J \tag{82.35}$$

and

$$(dK/dz) = -\chi_H H \tag{82.36}$$

are exact frequency-integrated moment equations. But to solve these equations one must know κ_J and χ_H, which implies solving the nongrey equations. An effective method sometimes used to handle transfer and energy balance in nongrey media is to rewrite (82.35) and (82.36) as

$$(dH/dz) = \kappa_P(B - k_J J) \tag{82.37}$$

and

$$(dK/dz) = -\chi_R k_H H \tag{82.38}$$

where the ratios $k_J \equiv (\kappa_J/\kappa_P)$ and $k_H \equiv (\chi_H/\chi_R)$ are to be determined iteratively, starting from an initial estimate of unity (or values from the previous time-step in a dynamical calculation—see §7.3).

The idea is to use (82.37) and (82.38) along with a constraint of energy balance to determine the temperature distribution, and then perform a frequency-by-frequency formal solution of the transfer equation, using the new temperature distribution, to update (J_ν/J) and (H_ν/H), and then k_J and k_H. With these improved estimates of k_J and k_H we can repeat the first step to determine an improved temperature distribution. Each step of this iteration procedure is relatively cheap. Nevertheless it presupposes that frequency-dependent opacities $\kappa_\nu(\rho, T)$ are available (which may not be true), and that we are willing to calculate the full frequency spectrum of the radiation field; in practice it may prove too costly in dynamical calculations for hundreds or thousands of timesteps. We then have little choice but to adopt $k_J \equiv 1$ and $k_H \equiv 1$; although the results so obtained are not exact, because in general κ_P is not the correct absorption average nor does χ_R equal χ_H except in the diffusion regime, they provide nonetheless a reasonable first approximation.

MULTIGROUP METHODS

The preceding discussion points out that detailed simulation of the opacity spectrum by the direct approach is generally too costly, while replacing it by one or two representative means may be too crude; we therefore seek a middle ground. The weakness of the mean opacity approach is that it averages over the entire spectrum. Given large fluctuations in κ_ν and the possibility of large differences between B_ν and J_ν, it is obvious that κ_P, say, is unlikely to equal κ_J exactly. However it is much easier to define a meaningful opacity in a narrow spectral range, and one asks whether it is possible to represent the physically important features of the opacity spectrum with a few (but more than one or two) astutely chosen parameters. Two such methods have been suggested and widely applied.

In the first, the *multigroup method*, the spectrum is divided into a number of *frequency groups*, each of which spans a definite range (ν_g, ν_{g+1}). Within each group, source terms and radiation quantities are replaced by values that can be viewed either as integrals over the group, for example,

$$B_g \equiv \int_{\nu_g}^{\nu_{g+1}} B_\nu \, d\nu, \tag{82.39}$$

$$J_g \equiv \int_{\nu_g}^{\nu_{g+1}} J_\nu \, d\nu, \tag{82.40}$$

(and similarly for H_g and K_g), or as representative constant values within the group. The zeroth moment equation for group g in a static LTE medium is then

$$(dH_g/dz) = \kappa_{P,g} B_g - \bar{\kappa}_g J_g, \tag{82.41}$$

where the *group Planck mean* is

$$\kappa_{P,g} \equiv \int_{\nu_g}^{\nu_{g+1}} B_\nu \, d\nu / B_g. \tag{82.42}$$

We must now decide what value is to be assigned to $\bar{\kappa}_g$ to obtain the correct total absorption in the group. If we were to take literally the picture that J_g is constant within (ν_g, ν_{g+1}), then we should use the straight average opacity

$$\bar{\kappa}_g \equiv \int_{\nu_g}^{\nu_{g+1}} \kappa_\nu \, d\nu / (\nu_{g+1} - \nu_g). \tag{82.43}$$

However, if we use (82.43), then at great depth where $J_\nu \rightarrow B_\nu$, we do not necessarily recover equality between the energy absorbed and emitted within the group (i.e., between $\bar{\kappa}_g J_g$ and $\kappa_{P,g} B_g$) even though $J_g \equiv B_g$, because we have used different weighting schemes in (82.42) and (82.43). Furthermore, in the limit of using only one group, (82.43) is not at all reasonable physically. For these reasons it is usually argued that the multigroup zeroth moment equations should be written as

$$(dH_g/dz) = \kappa_{P,g}(B_g - J_g), \tag{82.44}$$

which reduces to the mean-opacity method for a single group. On the other hand, in the optically thin limit there is no rationale for weighting the absorption term by the Planck function, that is, for replacing $\bar{\kappa}_g$ with $\kappa_{P,g}$, and it may actually be preferable to use $\bar{\kappa}_g$ as defined in (82.43) (e.g., in an optically thin layer illuminated by a perfectly smooth continuum having a radiation temperature markedly different from the local material temperature).

By similar reasoning, the first moment equation for group g can be written

$$(dK_g/dz) = -\bar{\chi}_g H_g. \tag{82.45}$$

Here one argues that to obtain the correct total flux transport we should use a harmonic mean for $\bar{\chi}_g$. Again if we literally take K_g to be constant in a group we would use the simple harmonic mean

$$(\bar{\chi}_g)^{-1} \equiv \int_{\nu_g}^{\nu_{g+1}} \chi_\nu^{-1}\, d\nu/(\nu_{g+1} - \nu_g). \tag{82.46}$$

But this choice of $\bar{\chi}_g$ does not necessarily yield the correct flux in the diffusion limit; to guarantee that we should write

$$(dK_g/dz) = -\chi_{R,g} H_g \tag{82.47}$$

where $\chi_{R,g}$ is the *group Rosseland mean*

$$(\chi_{R,g})^{-1} \equiv \int_{\nu_g}^{\nu_{g+1}} \chi_\nu^{-1}(\partial B_\nu/\partial T)\, d\nu \bigg/ \int_{\nu_g}^{\nu_{g+1}} (\partial B_\nu/\partial T)\, d\nu. \tag{82.48}$$

In most multigroup formulations, (82.47) is used in preference to (82.45). Like (82.44) for the zeroth moment, (82.47) provides a reasonable representation in the limit of one group; (82.45) does not, but may be more realistic for optically thin material.

In summary, the formulation of the multigroup method is not unique, and the results are unavoidably somewhat ambiguous. The method is least accurate for coarse frequency groups, for which there can be significant differences among the various group means, leading to the same problems as in the mean-opacity method. It gives increasingly better results as more, and finer, groups are used, because then the details of the weighting procedure are less important and the various group averages become more nearly equal; of course it also becomes more costly.

OPACITY DISTRIBUTION FUNCTIONS

The second important technique for handling radiative transfer in complex spectra uses *opacity distribution functions* (ODF). The ODF method has been applied extensively in astrophysics and yields excellent results in a wide variety of situations (**C3**), (**D1**). Here the whole spectrum is divided into a number of frequency *intervals*. For a calculation of the total radiative energy and momentum transport in each interval, the exact position of a particular feature (e.g., a spectral line) within the interval is not important; instead, we need to know the fraction of the interval that is relatively transparent (continuum), moderately opaque (weak lines), or very opaque (strong lines). Therefore in each interval we compute the opacity at a large number of uniformly spaced points and bin similar opacity values; we thus construct a distribution function giving the value of the opacity versus the cumulative fraction of the interval covered by opacity less than or equal to this value—see (**K3,** 3–7) or (**M2,** 167–169). This function is smooth and is well represented by a relatively small number of *pickets*, each of which covers a prechosen fraction of the interval with a constant opacity equal to the average of the distribution function in that band; an example is shown in (**K3,** 7).

For each picket we write a transfer equation (or moment equations); in these equations the opacity now has a unique value, which eliminates the ambiguity of the multigroup method. In this respect the ODF is similar to the direct method, and can be viewed as a way of degrading unwanted details of the opacity spectrum (which act as high-frequency "noise") to the minimum level required to give correct energy transport. (The multigroup method did just the opposite: it reduced the overly severe filtering inherent in whole-spectrum means by applying the averaging scheme over smaller intervals.)

THE OPACITY SAMPLING TECHNIQUE

We just mention one other approach: the *opacity sampling technique* (**S3**). Here one solves the transfer equation using actual opacities at a large number of frequencies chosen at random throughout the spectrum. This method has been used successfully to construct static model atmospheres, but it does not seem easily adaptable to dynamical problems so we will not discuss it further.

83. *Numerical Methods*

Analytical solutions of transfer problems are rare, and in most cases of interest we must use numerical methods. We describe here an approach that has proven to be general, flexible, and powerful in treating both radiative transfer and its coupling to the constraints of energy and momentum balance and to the equations of statistical equilibrium. For the present we consider only time-independent transfer in static media; we extend the method to other cases in later chapters.

THE PROBLEM OF SCATTERING

An important obstacle encountered in solving transfer problems is the scattering term, which decouples the radiation field from local sources and sinks, and introduces global transport of photons over large distances. This term permits an open boundary to make itself felt at great depth ($\tau_\nu \gg 1$) in the medium, and allows J_ν to depart significantly from B_ν even at depths where one would have expected them to be identical.

Scattering terms may appear explicitly in the source function. For example, S_ν may have the form [cf. (77.8)]

$$S_\nu = \xi_\nu B_\nu + (1-\xi_\nu)J_\nu \tag{83.1}$$

where ξ_ν is the *thermal coupling parameter*

$$\xi_\nu = \kappa_\nu/(\kappa_\nu + \sigma_\nu). \tag{83.2}$$

In astrophysical media, ξ_ν can sometimes be quite small. For instance, in hot stellar atmospheres Thomson scattering can be the dominant opacity source, and ξ_ν may be of order 10^{-4} into very deep layers of the atmosphere. In spectral lines, the thermalization parameter ε in (77.11) can be very small, and because $r \ll 1$, ξ_ν will also be small, say 10^{-8} or even less.

To see the implications of dominant scattering terms, consider the following simplified problem. Suppose the depth variation of the Planck function is

$$B_\nu = a_\nu + b_\nu \tau_\nu, \tag{83.3}$$

and that ξ_ν is depth independent. Using (83.1) in the zeroth moment equation (78.15) we have

$$(\partial H_\nu/\partial\tau_\nu) = \xi_\nu(J_\nu - B_\nu), \tag{83.4}$$

and the first-order moment equation is

$$(\partial K_\nu/\partial\tau_\nu) = H_\nu. \tag{83.5}$$

Making the Eddington approximation $K_\nu = \frac{1}{3}J_\nu$ we can combine (83.4) and (83.5) into

$$\tfrac{1}{3}(\partial^2 J_\nu/\partial\tau_\nu^2) = \xi_\nu(J_\nu - B_\nu). \tag{83.6}$$

Because $(\partial^2 B_\nu/\partial\tau_\nu^2)$ from (83.3) is zero, we can replace J_ν in the second derivative by $(J_\nu - B_\nu)$. Solving, we obtain

$$J_\nu = B_\nu + \alpha_\nu \exp[-(3\xi_\nu)^{1/2}\tau_\nu] + \beta_\nu \exp[(3\xi_\nu)^{1/2}\tau_\nu]. \tag{83.7}$$

The unknown constants α_ν and β_ν are determined by boundary conditions. At great depth we demand $J_\nu \rightarrow B_\nu$, hence $\beta_\nu \equiv 0$. At the surface we use the Eddington–Krook boundary condition $J_\nu(0) = \sqrt{3}\,H_\nu(0)$ (which is consistent with the exact solution of the grey problem). But from (83.5) we can also write $H_\nu(0) = \frac{1}{3}(\partial J_\nu/\partial\tau_\nu)_0$; evaluating the derivative from (83.7) and applying the boundary condition we find α_ν, and thus obtain finally

$$J_\nu(\tau_\nu) = a_\nu + b_\nu\tau_\nu + (b_\nu - \sqrt{3}\,a_\nu)\exp[-(3\xi_\nu)^{1/2}\tau_\nu]/[\sqrt{3}\,(1+\xi_\nu^{1/2})]. \tag{83.8}$$

Equation (83.8) reveals the essential physics of the problem. For simplicity consider an isothermal medium ($b_\nu \equiv 0$). First, we see that J_ν can depart markedly from B_ν at the surface; $J_\nu(0) = \xi_\nu^{1/2}B_\nu/(1+\xi_\nu^{1/2}) \approx \xi_\nu^{1/2}B_\nu$ for $\xi_\nu \ll 1$. Thus for $\xi_\nu \ll 1$, $J_\nu(0) \ll B_\nu(0)$. Second, this departure extends to great depths in the medium. The slow decay of the exponential in (83.8) implies that $J_\nu \rightarrow B_\nu$ only when $\tau_\nu \gtrsim (\xi_\nu)^{-1/2}$; the values ξ_ν quoted above show that this *thermalization depth* can be orders of magnitude larger than $\tau_\nu \sim 1$.

We can easily understand why the thermalization depth is so large from physical considerations. The parameter ξ_ν is essentially the probability per scattering event that a photon is thermally destroyed. To assure thermalization, the photon must scatter on the order of $n \sim 1/\xi_\nu$ times. Because the photon executes a random walk, in n scatterings it will travel $n^{1/2} \sim \xi_\nu^{-1/2}$ mean free paths in any direction, in particular toward the boundary surface. Thus when the point of photon emission is at a depth $\tau_\nu \gtrsim \xi_\nu^{-1/2}$ the probability that the photon will be thermalized before it escapes approaches unity, hence $J_\nu \to B_\nu$.

These results have important implications for developing numerical methods that can handle scattering terms successfully. For example, suppose that instead of solving the problem as we did above, we decided to compute $J_\nu = \Lambda(S_\nu)$; because S_ν contains J_ν, this is an integral equation. To avoid solving the integral equation directly we might try to proceed by *iteration*. Suppose we take $J_\nu = B_\nu$ as a first guess. Using $S_\nu = B_\nu$ we compute a new estimate of J_ν; using this J_ν we then calculate a new value of S_ν, re-evaluate the lambda operator for J_ν, and iterate. We saw in §79 that the kernel of Λ has an exponential falloff, which implies an effective information-propagation range of only $\Delta\tau \sim 1$; in the present problem one would therefore need to perform at least $\xi_\nu^{-1/2}$ iterations (a large number!) to propagate information about the existence of the surface [and the large departure of $J_\nu(0)$ from $B_\nu(0)$] over the entire thermalization depth. This failure also applies to all other equivalent methods that attempt to find S_ν by an iteration like the one just described (which we refer to generically as "Λ iteration").

We were able to obtain the correct solution (83.8) because we solved the problem by an analytical method that dealt *explicitly* with the scattering term. We conclude that any successful numerical method must do likewise, that is, scattering terms must appear explicitly in the source function, and the method must solve the resulting equations directly. In fact, the situation can be even more complex than we have indicated because scattering terms can be "hidden" in other aspects of the problem. For example, if we demand radiative equilibrium, the requirement that $\int \chi_\nu S_\nu \, d\nu = \int \chi_\nu J_\nu \, d\nu$ imposes a coupling of S_ν to J_ν that implies that S_ν will behave like a scattering term; this is seen clearly in the grey problem where $S \equiv J$, and the transfer equation (82.5) is the equivalent of a *pure scattering problem* (i.e., $\xi_\nu \equiv 0$). Furthermore, we will see in §87 that when we drop the assumption of LTE, the equations of statistical equilibrium also imply the presence of scattering terms in the source function.

The discussion leading to (83.7) and (83.8) also shows that in solving transfer problems we must deal with the *two-point boundary conditions* posed by (79.2) to (79.4); the same will be true for any numerical method. One might try to evade this problem by integrating the transfer equation from, say, the deepest point in the medium to the surface, using as initial conditions a guess for $I^-(\tau_{\max}, \mu)$ at the lower boundary, along with $I^+(\tau_{\max}, \mu)$ given by the lower boundary condition. However, unless the guess for I^- is perfect, the values computed for $I^-(0, \mu)$ will not agree with those specified by the upper boundary condition. One must then try to make a new guess for $I^-(\tau_{\max})$ that would yield improved agreement between the computed and imposed surface values of I^-.

Not only is this *eigenvalue method* [$I^-(\tau_{\max}, \mu_i)$ is to be determined for all $\mu_i < 0$] inefficient, it is also strongly unstable. Note that the solution (83.7) contains both ascending and descending exponentials; the latter is the true solution, the former is a parasite. Unless we suppress the parasite [as we did analytically in deriving (83.8)], it tends to run away and swamp the true solution. To avoid such problems, our adopted numerical method must explicitly treat the two-point boundary-value nature of the transfer problem from the outset [see (**M2,** 150–151) for further discussion].

SECOND-ORDER FORM OF THE TRANSFER EQUATION

A powerful computational method for solving transfer problems is based on a second-order form of the transfer equation. Consider a static, planar medium. Choose the column mass $dm = -\rho\, dz$ or

$$m(z) = \int_z^{z_{\max}} \rho(z')\, dz' \tag{83.9}$$

as independent variable; this is a Lagrangean variable suitable for dynamical calculations. Note that m increases downward into the medium; the opposite convention is used in stellar structure calculations where M_r increases outward with radius, cf. §80. Let

$$\omega(\nu) \equiv \chi(\nu)/\rho \tag{83.10}$$

be the opacity per gram, in terms of which we write optical depth increments as $d\tau_\nu = \omega_\nu\, dm$.

For radiation moving in two antiparallel pencils $\pm\mu$, we have two transfer equations

$$\pm\mu[\partial I(\pm\mu, \nu)/\partial m] = \omega(\nu)[I(\pm\mu, \nu) - S(\nu)], \tag{83.11}$$

where for brevity we suppress explicit mention of m dependence. As in §81 define a mean-intensity-like variable

$$j(\mu, \nu) \equiv \tfrac{1}{2}[I(+\mu, \nu) + I(-\mu, \nu)], \qquad (0 \le \mu \le 1), \tag{83.12}$$

and a fluxlike variable

$$h(\mu, \nu) \equiv \tfrac{1}{2}[I(+\mu, \nu) - I(-\mu, \nu)], \qquad (0 \le \mu \le 1). \tag{83.13}$$

Then adding the two equations (83.11) for $\pm\mu$, we obtain

$$\mu[\partial h(\mu, \nu)/\partial m] = \omega(\nu)[j(\mu, \nu) - S(\nu)], \tag{83.14}$$

and subtracting them we have

$$\mu[\partial j(\mu, \nu)/\partial m] = \omega(\nu) h(\mu, \nu). \tag{83.15}$$

These equations, first derived by P. Feautrier (**F1**), strongly resemble the zeroth and first moments of the transfer equation, but with two important differences: (1) they contain only j and h (i.e., the system closes), and (2) they are angle dependent.

Using (83.15) we can eliminate h from (83.14), obtaining the second-order equation

$$\frac{\mu^2}{\omega_\nu}\frac{\partial}{\partial m}\left[\frac{1}{\omega_\nu}\frac{\partial j(\mu, \nu)}{\partial m}\right] = j(\mu, \nu) - S(\nu). \tag{83.16}$$

or, in abbreviated notation,

$$\mu^2(\partial^2 j_{\mu\nu}/\partial\tau_\nu^2) = j_{\mu\nu} - S_\nu, \qquad (0 \le \mu \le 1). \tag{83.17}$$

These equations, supplemented by boundary conditions, can be solved by efficient numerical algorithms to be discussed shortly. We stress that in solving (83.16) and (83.17), any scattering terms in S_ν are to be written out *explicitly*; this introduces integrals of $j_{\mu\nu}$ over angle (and sometimes over frequency) on the right-hand side. Having solved (83.17) for $j(\mu, \nu)$ we can find $h(\mu, \nu)$ from (83.15).

An important property of (83.14) to (83.17) is that they are accurate in the diffusion regime. Thus if we integrate (83.17) over μ we find

$$J_\nu = S_\nu + (\partial^2 K_\nu/\partial\tau_\nu^2), \tag{83.18}$$

which in the diffusion regime yields

$$J_\nu = B_\nu + \tfrac{1}{3}(\partial^2 B_\nu/\partial\tau_\nu^2), \tag{83.19}$$

agreeing with (80.3) to second order. Similarly the integral over angle of μ times (83.15) yields the first term of (80.4) in the diffusion limit; as the next term is of third order, this implies the flux is accurate to second order. Experience shows that (83.14) to (83.17) behave correctly at both small and large optical depths.

BOUNDARY CONDITIONS

To obtain a unique solution of (83.17) we must impose boundary conditions at $\tau = 0$ and $\tau = \tau_{\max}$. At $\tau_\nu = 0$, we usually set $I(-\mu, \nu) \equiv 0$; hence from (83.12) and (83.13) $h_{\mu\nu}(0) \equiv j_{\mu\nu}(0)$, and (83.15) then yields

$$\mu(\partial j_{\mu\nu}/\partial\tau_\nu)_0 = j_{\mu\nu}(0). \tag{83.20}$$

In a finite slab, at $\tau = \tau_{\max}$ we specify $I(\tau_{\max}, +\mu, \nu) = I^+_{\mu\nu}$; from the identity $h_{\mu\nu}(\tau_{\max}) \equiv I^+_{\mu\nu} - j_{\mu\nu}(\tau_{\max})$, (83.15) then becomes

$$\mu(\partial j_{\mu\nu}/\partial\tau_\nu)_{\tau_{\max}} = I^+_{\mu\nu} - j_{\mu\nu}(\tau_{\max}). \tag{83.21}$$

Equations (83.17), (83.20), and (83.21) are sufficient to specify the run of $j_{\mu\nu}$ with depth in the slab.

In a semi-infinite medium we can impose the diffusion approximation at $\tau_{\max}$ and take $I(\tau_{\max}, \mu, \nu) = B_\nu(\tau_{\max}) + \mu(\partial B_\nu/\partial\tau_\nu)_{\tau_{\max}}$, whence $j_{\mu\nu} = B_\nu(\tau_{\max})$

and $h_{\mu\nu}(\tau_{\max}) = \mu(\partial B_\nu/\partial\tau_\nu)_{\tau_{\max}}$; (83.21) then reduces to

$$(\partial j_{\mu\nu}/\partial\tau_\nu)_{\tau_{\max}} = (\partial B_\nu/\partial\tau_\nu)_{\tau_{\max}} = [(\partial B_\nu/\partial T)(dT/dm)/\omega_\nu]_{\tau_{\max}}. \tag{83.22}$$

Alternatively we may wish to specify the total flux H transported across the lower boundary. In the diffusion limit we have

$$H = \int_0^\infty d\nu \int_0^1 d\mu\, \mu h_{\mu\nu} = (dT/dm) \int_0^\infty d\nu\omega_\nu^{-1}(\partial B_\nu/\partial T) \int_0^1 d\mu\, \mu^2, \tag{83.23a}$$

or

$$(dT/dm) = 3H \Big/ \int_0^\infty \omega_\nu^{-1}(\partial B_\nu/\partial T)\, d\nu; \tag{83.23b}$$

hence from (83.22)

$$\left.\frac{\partial j_{\mu\nu}}{\partial\tau_\nu}\right|_{\tau_{\max}} = \left[\frac{3H}{\omega_\nu}\left(\frac{\partial B_\nu}{\partial T}\right) \Big/ \int_0^\infty \frac{1}{\omega_\nu}\frac{\partial B_\nu}{\partial T}\, d\nu\right]_{\tau_{\max}} \tag{83.24}$$

Equations (83.17), (83.20), and either (83.22) or (83.24) are sufficient to specify the run of $j_{\mu\nu}$ with depth in a semi-infinite medium.

DISCRETIZATION

We now convert the differential equation (83.17) and its accompanying boundary conditions into *difference equations* by a discretization of all variables. We choose a discrete set of angle points $\{\mu_m\}$, $(m = 1, \ldots, M)$, and frequency points $\{\nu_n\}$, $(n = 1, \ldots, N)$, spanning the ranges $0 \le \mu \le 1$ and $0 \le \nu \le \infty$. We divide the medium into a set of D mass shells whose boundaries are specified by the mesh $\{m_d\}$, $(d = 1, \ldots, D+1)$; each cell has a mass $m_{d+(1/2)} = m_{d+1} - m_d$. In general, the mass cells will be of unequal size. Variables whose values are specified at cell centers are given half-integral indices [e.g., $j_{d+(1/2),mn} \equiv j(m_{d+(1/2)}, \mu_m, \nu_n)$]. Variables specified on cell surfaces are given integer indices [e.g., $h_{dmn} \equiv h(m_d, \mu_m, \nu_n)$]; see Figure 83.1.

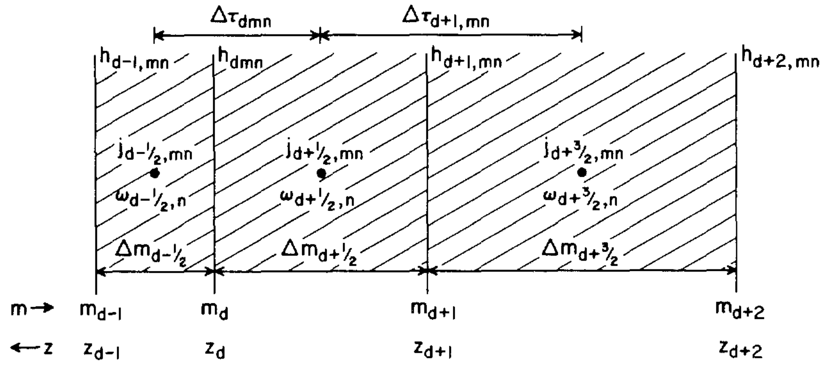

Fig. 83.1 Centering of radiation variables on Lagrangean mesh.

Integrals are replaced by quadrature sums, for example,

$$\int_0^\infty y(\nu)\,d\nu \to \sum_{n=1}^{N} a_n y(\nu_n) \tag{83.25a}$$

and

$$\int_0^1 y(\mu)\,d\mu \to \sum_{m=1}^{M} b_m y(\mu_m). \tag{83.25b}$$

For example, source functions such as (77.8) and (77.11) are represented as

$$S_{d+(1/2),n} = \alpha_{d+(1/2),n} \sum_{k=1}^{K} w_k j_{d+(1/2),k} + \beta_{d+(1/2),n}. \tag{83.26}$$

The first term in (83.26) is the scattering integral, and the second is the thermal term. Here we have grouped all combinations of angles and frequencies into a single serial set with index $k = 1, \ldots, K = MN$ [e.g., $(\mu_k, \nu_k) \equiv (\mu_m, \nu_n)$ where $k = m + (n-1)M$], and w_k is a combined weight for both angle and frequency quadratures, which may include a profile function.

Similarly, derivatives are replaced by difference formulae, for example,

$$(d^2x/d\tau^2)_{d+(1/2)} = [(dx/d\tau)_{d+1} - (dx/d\tau)_d]/\Delta\tau_{d+(1/2)}, \tag{83.27}$$

where

$$(dx/d\tau)_d = [x_{d+(1/2)} - x_{d-(1/2)}]/\Delta\tau_d. \tag{83.28}$$

Thus defining

$$\Delta\tau_{dk} \equiv \tfrac{1}{2}[\omega_{d-(1/2),k}\,\Delta m_{d-(1/2)} + \omega_{d+(1/2),k}\,\Delta m_{d+(1/2)}], \tag{83.29}$$

and

$$\Delta\tau_{d+(1/2),k} \equiv \tfrac{1}{2}(\Delta\tau_{dk} + \Delta\tau_{d+1,k}), \tag{83.30}$$

a second-order accurate difference representation of (83.17) is

$$\begin{aligned} \frac{1}{\Delta\tau_{d+(1/2),k}} \Bigg[\left(\frac{\mu_k^2}{\Delta\tau_{d+1,k}}\right) j_{d+(3/2),k} - \mu_k^2 \left(\frac{1}{\Delta\tau_{dk}} + \frac{1}{\Delta\tau_{d+1,k}}\right) j_{d+(1/2),k} \\ + \left(\frac{\mu_k^2}{\Delta\tau_{dk}}\right) j_{d-(1/2),k} \Bigg] = j_{d+(1/2),k} - S_{d+(1/2),k}, \end{aligned} \tag{83.31}$$

$$(k = 1, \ldots, K),\ (d = 2, \ldots, D-1)$$

where $S_{d+(1/2),k}$ is given by (83.26). [Fourth-order accurate difference equations can be written using Hermite integration formulae (**A6**).] These $(D-2)$ sets of equations must be augmented by two sets of boundary conditions.

Consider first the upper boundary. From (83.14) we have

$$h_{2k} = h_{1k} + [\Delta\tau_{(3/2)k}/\mu_k][j_{(3/2)k} - S_{(3/2)k}], \tag{83.32}$$

where $\Delta\tau_{(3/2)k} \equiv \omega_{(3/2)k}\,\Delta m_{3/2}$. From (83.15),

$$h_{2k} = \mu_k[j_{(5/2)k} - j_{(3/2)k}]/\Delta\tau_{2k}. \tag{83.33}$$

Applying (83.20) from the surface of the uppermost cell to its center we have

$$j_{1k} \equiv h_{1k} = \mu_k[j_{(3/2)k} - j_{1k}]/\tfrac{1}{2}\,\Delta\tau_{(3/2)k}, \tag{83.34}$$

whence

$$j_{1k} = h_{1k} = j_{(3/2)k}/(1 + \tfrac{1}{2}\,\Delta\tau_{(3/2)k}/\mu_k). \tag{83.35}$$

Using (83.33) and (83.35) in (83.32) we obtain the desired boundary condition

$$\begin{aligned} \mu_k[j_{(5/2)k} - j_{(3/2)k}]/\Delta\tau_{2k} = j_{(3/2)k}/[1 + \tfrac{1}{2}\Delta\tau_{(3/2)k}/\mu_k] \\ + [\Delta\tau_{(3/2)k}/\mu_k][j_{(3/2)k} - S_{(3/2)k}]. \end{aligned} \tag{83.36}$$

To obtain a lower boundary condition, we can use (83.21), (83.22), or (83.24). In particular, (83.24) implies

$$\frac{[j_{D+(1/2),k} - j_{D-(1/2),k}]}{\Delta\tau_{Dk}} = \frac{3H}{\omega_{D+(1/2),k}} \left(\frac{\partial B_k}{\partial T}\right)_{D+(1/2)} \bigg/ \left[\sum_{n=1}^{N} \frac{a_n}{\omega_{D+(1/2),n}} \left(\frac{\partial B_n}{\partial T}\right)_{D+(1/2)}\right]. \tag{83.37}$$

Equation (83.37) is only of first-order accuracy, but this is usually sufficient in the diffusion regime. Using Hermite formulae it is possible to write third-order accurate boundary conditions (**A6**).

Equations (83.31), (83.36), and (83.37) comprise *DK* equations in the same number of unknowns $\{j_{d+(1/2),k}\}$; let us now consider how they are to be solved.

THE FORMAL SOLUTION

The simplest transfer problem, called the *formal solution*, is to calculate the radiation field at all depths, angles, and frequencies when the source function is known [e.g., in an LTE medium ($S_\nu \equiv B_\nu$)] whose temperature structure is given. The formal solution is also used to evaluate variable Eddington factors in certain iterative procedures discussed below.

Represent the depth variation of j for a particular angle and frequency (μ_k, ν_k) by the column vector

$$\mathbf{j}_k \equiv [j_{(3/2)k}, j_{(5/2)k}, \ldots, j_{D+(1/2),k}], \tag{83.38}$$

and the (known) depth variation of $S(\nu_k)$ by

$$\mathbf{S}_k \equiv [S_{(3/2)k}, S_{(5/2)k}, \ldots, S_{D+(1/2),k}]. \tag{83.39}$$

Then the transfer equation (83.31) and boundary conditions (83.36) and (83.37) for $j_{d+(1/2),k}$ $(d = 1, \ldots, D)$ are of the form

$$\mathbf{T}_k \mathbf{j}_k = \mathbf{S}_k, \tag{83.40}$$

where $\mathbf{T}_k$ is a $(D \times D)$ *tridiagonal* matrix.

The solution of (83.40) is effected by a standard Gaussian elimination scheme [cf. (83.52) to (83.54)], which requires of the order of cD operations, where c is a numerical constant of order unity. If we have a total of K angle-frequency choices, the computational effort to obtain the full radiation field scales as $cDK = cDMN$, which is irreducible because we wish to determine DK values of j. On vector computers, several systems of the form (83.40) can be solved in parallel, resulting in an enormous increase in efficiency.

Having determined $j_{d+(1/2),k}$ for all d and k we can calculate the moments

$$J_{d+(1/2),n} \equiv \sum_m b_m j_{d+(1/2),mn} \tag{83.41}$$

and

$$K_{d+(1/2),n} = \sum_m b_m \mu_m^2 j_{d+(1/2),mn}, \tag{83.42}$$

and hence the Eddington factor $f_{d+(1/2),n} \equiv [K_{d+(1/2),n}/J_{d+(1/2),n}]$ for $d = 1, \ldots, D$. Using the difference representation of (83.15) we can calculate h_{dk} and hence fluxes on the cell boundaries

$$H_{dn} = \sum_m b_m \mu_m h_{dmn}, \tag{83.43}$$

for $d = 2, \ldots, D$. The flux at the lower boundary is fixed by the lower boundary condition. At the upper boundary we calculate j_{1k} using (83.35), and can then evaluate

$$J_{1n} = \sum_m b_m j_{1mn} \equiv \mathscr{j}_{1n} J_{(3/2)n}, \tag{83.44}$$

$$H_{1n} = \sum_m b_m \mu_m j_{1mn} \equiv \mathscr{h}_{1n} J_{1n}, \tag{83.45}$$

and

$$K_{1n} = \sum_m b_m \mu_m^2 j_{1mn} = f_{1n} J_{1n}. \tag{83.46}$$

The geometrical factors $\mathscr{j}_{1n}$ and $\mathscr{h}_{1n}$ will be used to pose boundary conditions for the radiation moment equations [see (83.61)].

THE FEAUTRIER METHOD

A more realistic transfer problem is to calculate the radiation field when the source function explicitly contains scattering terms as in (83.26). We solve such problems by two different methods. In the *Feautrier method* (**F1**), one defines vectors

$$\mathbf{j}_{d+(1/2)} \equiv [j_{d+(1/2),1}, j_{d+(1/2),2}, \ldots, j_{d+(1/2),K}], \qquad (d = 1, \ldots, D), \tag{83.47}$$

containing all angle-frequency components of the radiation field at a single depth, and vectors

$$\mathbf{L}_{d+(1/2)} \equiv [\beta_{d+(1/2),1}, \beta_{d+(1/2),2}, \ldots, \beta_{d+(1/2),K}], \qquad (d = 1, \ldots, D), \tag{83.48}$$

containing the angle-frequency components of the thermal source term.

The transfer equation (83.31) plus boundary conditions (83.36) and (83.37) can then be written as a *block tridiagonal* system of the form

$$-\mathbf{A}_{d+(1/2)}\mathbf{j}_{d-(1/2)}+\mathbf{B}_{d+(1/2)}\mathbf{j}_{d+(1/2)}-\mathbf{C}_{d+(1/2)}\mathbf{j}_{d+(3/2)} \\ =\mathbf{L}_{d+(1/2)}, \qquad (d=1,\ldots,D), \tag{83.49}$$

The matrices **A**, **B**, and **C** are all of dimension $(K\times K)$; **A** and **C** are diagonal, containing portions of the finite-difference operator for all angle-frequency points down the diagonal. **B** is a full matrix containing terms from the difference operator down the diagonal, plus diagonal and off-diagonal terms coupling radiation at each angle-frequency point to all others through the quadrature sum for the scattering integral in (83.26). [If Hermite formulae (**A6**) are used, **A** and **C** are also full.] The upper boundary condition implies that $\mathbf{A}_{3/2}\equiv 0$, and the lower boundary condition implies $\mathbf{C}_{D+(1/2)}\equiv 0$.

Equations (83.49) are solved by Gaussian elimination. At level $d+\frac{1}{2}$ we express $\mathbf{j}_{d+(1/2)}$ in terms of $\mathbf{j}_{d+(3/2)}$ and use this expression to eliminate $\mathbf{j}_{d+(1/2)}$ from the next equation. Starting at the upper boundary we have

$$\mathbf{j}_{3/2}=(\mathbf{B}_{3/2}^{-1}\mathbf{C}_{3/2})\mathbf{j}_{5/2}+(\mathbf{B}_{3/2}^{-1}\mathbf{L}_{3/2})\equiv\mathbf{D}_{3/2}\mathbf{j}_{5/2}+\boldsymbol{\nu}_{5/2}. \tag{83.50}$$

Substituting (83.50) into (83.49) for $d=2$ we find

$$\mathbf{j}_{5/2}=(\mathbf{B}_{5/2}-\mathbf{A}_{5/2}\mathbf{D}_{3/2})^{-1}\mathbf{C}_{5/2}\mathbf{j}_{7/2}+(\mathbf{B}_{5/2}-\mathbf{A}_{5/2}\mathbf{D}_{3/2})^{-1}(\mathbf{L}_{5/2}+\mathbf{A}_{5/2}\boldsymbol{\nu}_{3/2}) \\ \equiv\mathbf{D}_{5/2}\mathbf{j}_{7/2}+\boldsymbol{\nu}_{5/2}, \tag{83.51}$$

and in general we have

$$\mathbf{j}_{d+(1/2)}=\mathbf{D}_{d+(1/2)}\mathbf{j}_{d+(3/2)}+\boldsymbol{\nu}_{d+(1/2)}, \tag{83.52}$$

where

$$\mathbf{D}_{d+(1/2)}\equiv[\mathbf{B}_{d+(1/2)}-\mathbf{A}_{d+(1/2)}\mathbf{D}_{d-(1/2)}]^{-1}\mathbf{C}_{d+(1/2)}, \tag{83.53}$$

and

$$\boldsymbol{\nu}_{d+(1/2)}\equiv[\mathbf{B}_{d+(1/2)}-\mathbf{A}_{d+(1/2)}\mathbf{D}_{d-(1/2)}]^{-1}[\mathbf{L}_{d+(1/2)}+\mathbf{A}_{d+(1/2)}\boldsymbol{\nu}_{d-(1/2)}]. \tag{83.54}$$

We compute $\mathbf{D}_{d+(1/2)}$ and $\boldsymbol{\nu}_{d+(1/2)}$ for $d=1$ through $d=D-1$. At $d=D$, $\mathbf{C}_{D+(1/2)}=0$, and $\mathbf{j}_{D+(1/2)}\equiv\boldsymbol{\nu}_{D+(1/2)}$. Having found $\mathbf{j}_{D+(1/2)}$ we obtain all other $\mathbf{j}_{d+(1/2)}$ for $d=D-1, D-2, \ldots, 1$ from (83.52) by successive back substitution. From $\mathbf{j}_{d+(1/2)}$, one can evaluate $J_{d+(1/2),n}$ and $S_{d+(1/2),n}$ using appropriate angle (or angle-frequency) quadratures. The forward-backward sweep enforces the two-point boundary conditions, and the explicit appearance of scattering terms guarantees correct thermalization. The method is computationally robust. To estimate the computational effort, we note that a solution of a full linear system of order n requires $O(n^3)$ operations; there are D such systems, hence the total effort scales as $cDK^3=cDM^3N^3$.

VARIABLE EDDINGTON FACTORS

In the Feautrier method, the scaling of the computational effort as the cube of the product of the number of angles and frequencies is very unfavorable.

Clearly it is essential to eliminate any angle-frequency information about the radiation field that is not absolutely necessary. In this vein, we note that typical scattering kernels in static media (or in the fluid frame of moving media) are isotropic, hence only J_ν, not $j_{\mu\nu}$, enters. This simplification holds even when the transfer equation is coupled to constraints of energy and momentum balance, and to the statistical equilibrium equations (cf. §88). We therefore eliminate the superfluous angular information by using moments of the transfer equation, closing the system by use of variable Eddington factors.

Thus, integrating (83.17), (83.20), and (83.24) over angle we obtain

$$\partial^2(f_\nu J_\nu)/\partial\tau_\nu^2 = J_\nu - S_\nu, \tag{83.55}$$

$$[\partial(f_\nu J_\nu)/\partial\tau_\nu]_0 = H_\nu(0), \tag{83.56}$$

and

$$[\partial(f_\nu J_\nu)/\partial\tau_\nu]_{\tau_{\max}} = \left[(H/\omega_\nu)(\partial B_\nu/\partial T)\Big/\int_0^\infty \omega_\nu^{-1}(\partial B_\nu/\partial T)\,d\nu\right]_{\tau_{\max}}. \tag{83.57}$$

The finite difference form of (83.55) is

$$\begin{aligned}\frac{1}{\Delta\tau_{d+(1/2),n}}&\left[\frac{f_{d-(1/2),n}J_{d-(1/2),n}}{\Delta\tau_{dn}} - \left(\frac{1}{\Delta\tau_{dn}}+\frac{1}{\Delta\tau_{d+1,n}}\right) f_{d+(1/2),n}J_{d+(1/2),n}\right.\\ &\left.+\frac{f_{d+(3/2),n}J_{d+(3/2),n}}{\Delta\tau_{d+1,n}}\right] = J_{d+(1/2),n} - S_{d+(1/2),n}, \qquad (d=2,\ldots,D-1).\end{aligned} \tag{83.58}$$

The lower boundary condition can be represented as

$$\begin{aligned}&\frac{f_{D+(1/2),n}J_{D+(1/2),n} - f_{D-(1/2),n}J_{D-(1/2),n}}{\Delta\tau_{Dn}} =\\ &\qquad\frac{3H}{\omega_{D+(1/2),n}}\left(\frac{\partial B_n}{\partial T}\right)_{D+(1/2)}\Big/\sum_n \frac{a_n}{\omega_{D+(1/2),n}}\left(\frac{\partial B_n}{\partial T}\right)_{D+(1/2)}.\end{aligned} \tag{83.59}$$

By an analysis similar to that leading from (83.32) to (83.36) one finds that

$$H_{1n} = \hbar_{1n}J_{1n} = \hbar_{1n}f_{(3/2)n}J_{(3/2)n}/(f_{1n}+\tfrac{1}{2}\Delta\tau_{(3/2)n}\hbar_{1n}); \tag{83.60}$$

hence the upper boundary condition can be written as

$$\begin{aligned}[f_{(5/2)n}J_{(5/2)n} - f_{(3/2)n}J_{(3/2)n}]/\Delta\tau_{2n} = {}&\hbar_{1n}f_{(3/2)n}J_{(3/2)n}/[f_{1n}+\tfrac{1}{2}\Delta\tau_{(3/2)n}\hbar_{1n}]\\ &+\Delta\tau_{(3/2)n}[J_{(3/2)n}-S_{(3/2)n}].\end{aligned} \tag{83.61}$$

Equations (83.58) to (83.61) are of the same form as (83.49) and are solved by the same Gaussian elimination scheme. But now the computational effort scales as cDN^3, lower by a factor of M^3; in typical calculations $M=3$ or 4.

To solve (83.58) to (83.61) we must know f_ν at all depths and $\hbar_{1\nu}$ at the surface. We proceed iteratively: (1) From a first guess for S_ν (e.g., $S_\nu = B_\nu$)

we carry out a formal solution, which yields $j_{\mu\nu}$ at all angles, frequencies, and depths. (2) We then calculate f_ν at all depths from (83.41) and (83.42), and $\mathscr{j}_{1\nu}$ and $\mathscr{h}_{1\nu}$ from (83.44) and (83.45). The essential point is that the Eddington factors are determined with substantially better accuracy than the radiation field itself because they are *shape factors* that depend only on the *ratio* of radiation moments. For instance, local scale-factor errors in J_ν and K_ν simply drop out. (3) Given f_ν and $\mathscr{h}_{1\nu}$, we solve (83.58) to (83.61) using expressions for S_ν in which scattering terms appear explicitly. In this step, one obtains the correct thermalization properties of J_ν. (4) We then re-evaluate S_ν using the new value of J_ν; this updated S_ν will, in general, differ from the original S_ν. We therefore recalculate new f_ν's via step (1), and iterate to convergence.

If I iterations are required to achieve convergence, the total computing effort scales as $I(cDMN + c'DN^3) \ll c''DM^3N^3$ for moderate values of M and I. Experience shows that the Eddington-factor iteration generally converges rapidly ($I \sim 3$), so that substantial savings are realized.

THE RYBICKI METHOD

An alternative to the Feautrier method was devised by Rybicki (**R1**). In Feautrier's method, all frequency-dependent information is grouped together at each depth, and the solution proceeds depth by depth; this method can handle an explicit frequency dependence of the scattering terms, such as those that arise in partial redistribution problems. But in many problems the scattering term is independent of frequency; for example, in line formation with complete redistribution only the quantity $\bar{J} \equiv \int \phi_\nu J_\nu \, d\nu$ appears. In such cases the frequency information retained by the Feautrier method is redundant, and Rybicki showed that the system can be reorganized in a way that has more favorable computing-time requirements.

Assume that the source function has the form $S_\nu = \alpha_\nu \bar{J} + \beta_\nu$, and let

$$\bar{\mathbf{J}} \equiv [\bar{J}_{3/2}, \bar{J}_{5/2}, \ldots, \bar{J}_{D+(1/2)}] \tag{83.62}$$

represent the run of $\bar{J}$ with depth. Then at each angle-frequency point k the transfer equation has the form [cf. (83.40)]

$$\mathbf{T}_k \mathbf{j}_k + \mathbf{U}_k \bar{\mathbf{J}} = \mathbf{K}_k, \qquad (k = 1, \ldots, K), \tag{83.63}$$

where $\mathbf{T}_k$ is a $(D \times D)$ tridiagonal matrix representing the differential operator, $\mathbf{U}_k$ is a $(D \times D)$ diagonal matrix containing the depth variation of the coefficient $\alpha_{d+(1/2),k}$ of the scattering term, and $\mathbf{K}_k$ is a vector of length D containing the depth variation of the thermal source term $\beta_{d+(1/2),k}$. In addition we have D equations that define $\bar{J}_{d+(1/2)}$, namely

$$\bar{J}_{d+(1/2)} = \sum_{k=1}^{K} w_{d+(1/2),k} j_{d+(1/2),k}, \qquad (d = 1, \ldots, D). \tag{83.64}$$

which are equivalent to the matrix equation

$$\bar{\mathbf{J}} = \sum_{k=1}^{K} \mathbf{V}_k \mathbf{j}_k \tag{83.65}$$

where each $\mathbf{V}_k$ is a $(D \times D)$ diagonal matrix containing the depth variation of the quadrature weight for angle-frequency point k in (83.64).

The system comprising (83.63) and (83.64) can be solved efficiently. For each k we solve the tridiagonal system (83.63) to find the vector $\mathbf{A}_k$ and the matrix $\mathbf{B}_k$ in

$$\mathbf{j}_k = \mathbf{T}_k^{-1}\mathbf{K}_k - (\mathbf{T}_k^{-1}\mathbf{U}_k)\bar{\mathbf{J}} \equiv \mathbf{A}_k - \mathbf{B}_k\bar{\mathbf{J}}. \tag{83.66}$$

Substituting (83.66) into (83.65) for all k we develop the final system $\mathbf{C}\bar{\mathbf{J}} = \mathbf{D}$, where $\mathbf{C}$ is the full $(D \times D)$ matrix

$$\mathbf{C} = \mathbf{I} + \sum_{k=1}^{K} \mathbf{V}_k \mathbf{B}_k, \tag{83.67}$$

and $\mathbf{D}$ is a vector of length D,

$$\mathbf{D} = \sum_{k=1}^{K} \mathbf{V}_k \mathbf{A}_k. \tag{83.68}$$

We solve the final system for $\bar{\mathbf{J}}$, from which we calculate $\mathbf{S}_k$, the run of the source function at angle-frequency point k over depth. One can then find any desired $\mathbf{j}_k$ from (83.66).

The calculation of each $\mathbf{B}_k$ requires $O(D^2)$ operations, as does the multiplication by $\mathbf{V}_k$ and summation into $\mathbf{C}$. Solution of the final full system requires $O(D^3)$ operations. Hence the total computing effort scales as $cD^2K + c'D^3 = cD^2MN + c'D^3$; in practice the first term usually dominates. We now see the advantage of the Rybicki scheme over the Feautrier scheme: the computing effort varies *linearly* with the number of angle-frequency points, rather than as the cube; therefore Rybicki's method is preferable for problems with large numbers of angles and frequencies. On the other hand, Rybicki's method works only if the scattering term is frequency independent, whereas Feautrier's method can handle partial redistribution. Furthermore, with Feautrier's scheme it is relatively easy to impose constraints of energy and momentum balance, and to couple the transfer equation to the equations of statistical equilibrium (cf. §88), whereas imposition of these additional constraints makes Rybicki's scheme prohibitively costly. One should analyze each problem to determine the relative cost of the two methods, and choose the one that is optimum [see (**M2,** 161) and (**M2,** §12–3)].

SPHERICAL GEOMETRY

While planar geometry is often adequate in astrophysical transfer problems, to study extended envelopes or the structure of a star as a whole we must work in spherical geometry. Using variable Eddington factors, one can write the transfer equation for spherical media in a form that closely resembles the planar equation.

On a radial optical depth scale $d\tau_\nu = -\chi_\nu \, dr$, the moment equations (78.18) and (78.19) are

$$\partial(r^2 H_\nu)/\partial\tau_\nu = r^2(J_\nu - S_\nu) \tag{83.69}$$

and

$$[\partial(f_\nu J_\nu)/\partial\tau_\nu] - (3f_\nu - 1)J_\nu/\chi_\nu r = H_\nu. \tag{83.70}$$

One cannot derive a simple second-order equation by substituting (83.70) for H_ν directly into (83.69). But we can cast the left-hand side of (83.70) into a more convenient form by introducing a *sphericality factor* q_ν defined (**A5**) such that

$$\frac{1}{q_\nu}\frac{\partial(f_\nu q_\nu J_\nu)}{\partial\tau_\nu} \equiv \frac{\partial(f_\nu J_\nu)}{\partial\tau_\nu} - \frac{(3f_\nu - 1)}{\chi_\nu r} J_\nu, \tag{83.71}$$

whence it follows that

$$\ln q_\nu = \int_{r_c}^{r} [(3f_\nu - 1)/r' f_\nu] \, dr', \tag{83.72}$$

where r_c is the *core radius*, that is, the inner boundary of the medium. Note that q_ν is a geometrical factor on the same footing as f_ν, and is determined if f_ν is given.

Using (83.71) we can rewrite (83.70) as

$$\partial(f_\nu q_\nu J_\nu)/\partial\tau_\nu = q_\nu H_\nu, \tag{83.73}$$

which, when substituted into (83.69) yields the *combined moment equation*

$$\frac{1}{q_\nu}\frac{\partial}{\partial\tau_\nu}\left[\frac{r^2}{q_\nu}\frac{\partial(f_\nu q_\nu J_\nu)}{\partial\tau_\nu}\right] = \frac{r^2}{q_\nu}(J_\nu - S_\nu). \tag{83.74}$$

Defining the new variable $dX_\nu \equiv (q_\nu/r^2)\, d\tau_\nu$, we rewrite (83.74) in the second-order form

$$\partial^2(f_\nu q_\nu J_\nu)/\partial X_\nu^2 = (r^4/q_\nu)(J_\nu - S_\nu). \tag{83.75}$$

To obtain an outer boundary condition at $r = R$, define the geometrical factor

$$\boldsymbol{h}_\nu \equiv H_\nu(R)/J_\nu(R) = \int_0^1 I(R, \mu, \nu)\mu \, d\mu \bigg/ \int_0^1 I(R, \mu, \nu)\, d\mu. \tag{83.76}$$

Then from (83.73) we have

$$\partial(f_\nu q_\nu J_\nu)/\partial X_\nu \,|_{r=R} = \boldsymbol{h}_\nu R^2 J_\nu(R). \tag{83.77}$$

At the inner boundary we apply the diffusion approximation, fixing the temperature gradient by demanding the correct total flux transport as in

(83.23) and (83.24). We then have

$$\left.\frac{\partial(f_\nu q_\nu J_\nu)}{\partial X_\nu}\right|_{r=r_c} = r_c^2 H_c \left[\frac{1}{\chi_\nu}\left(\frac{\partial B_\nu}{\partial T}\right) \Big/ \int_0^\infty \frac{1}{\chi_\nu}\left(\frac{\partial B_\nu}{\partial T}\right) d\nu\right]_{r=r_c}, \qquad (83.78)$$

where $H_c = L/(16\pi^2 r_c^2)$.

Equations (83.75), (83.77), and (83.78) can be discretized on a spherical mesh and solved by the Feautrier scheme. The mesh may represent either a set of surfaces of constant radii, $\{r_i\}$, $(i = 1, \ldots, I+1)$, with $r_1 = r_c$ and $r_{I+1} = R$, or a set of mass shells $\{M_i\}$, where M_i is the total mass contained inside r_i. These choices are convenient for Eulerian and Lagrangean calculations, respectively. In the latter, the radiation flux is placed on cell surfaces, and the mean intensity, radiation pressure, and material properties are located at cell centers. The computing effort to solve the system for N frequencies scales as cIN if there is no frequency coupling in the source function, and as cIN^3 if there is.

To carry out the computations just described we must know the Eddington factors. These are determined from a frequency-by-frequency formal solution for given values of S_ν using either of two methods. One method is to solve the transfer equation (76.9) along *rays* tangent to a set of spherical shells (see Figure 83.2). If s measures the path length from the symmetry axis along such a ray, it is easy to show that the differential operator $\mu(\partial/\partial r) + r^{-1}(1-\mu^2)(\partial/\partial\mu)$ is identically $(\partial/\partial s)$. Hence if $I^\pm$ denotes the intensity traveling along $\pm s$, (76.9) becomes

$$\pm[\partial I^\pm(s, p, \nu)/\partial s] = \chi(r, \nu)[S(r, \nu) - I^\pm(s, p, \nu)], \qquad (83.79)$$

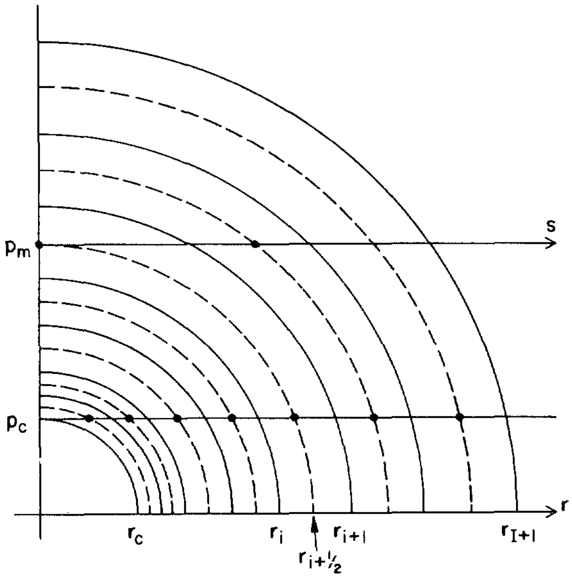

Fig. 83.2 Ray geometry in spherical symmetry.

where p is the impact parameter of the ray and $r = r(s, p) = (s^2 + p^2)^{1/2}$. If we define

$$d\tau(s, p, \nu) \equiv -\chi[r(s, p), \nu]\, ds, \tag{83.80}$$

$$j(s, p, \nu) \equiv \tfrac{1}{2}[I^+(s, p, \nu) + I^-(s, p, \nu)], \tag{83.81}$$

and

$$h(s, p, \nu) \equiv \tfrac{1}{2}[I^+(s, p, \nu) - I^-(s, p, \nu)]; \tag{83.82}$$

then the sum and difference of equations (83.79) yield

$$[\partial h(s, p, \nu)/\partial\tau(s, p, \nu)] = j(s, p, \nu) - S(s, p, \nu) \tag{83.83}$$

and

$$[\partial j(s, p, \nu)/\partial\tau(s, p, \nu)] = h(s, p, \nu), \tag{83.84}$$

hence

$$[\partial^2 j(s, p, \nu)/\partial\tau^2(s, p, \nu)] = j(s, p, \nu) - S(s, p, \nu). \tag{83.85}$$

To discretize we choose a set of rays defined by impact parameters $\{p_m\}$, $(m = 1, \ldots, M = I + C)$, where the first C rays intersect the core, with $p_C = r_1$, and the remainder are tangent to the spherical surfaces through cell centers [i.e., $p_m = r_{(m-C)+(1/2)}$]. The cell centers are defined such that $r_{i+(1/2)}$ contains half the volume (or mass) between r_i and r_{i+1}, that is,

$$r^3_{i+(1/2)} \equiv \tfrac{1}{2}(r_i^3 + r_{i+1}^3). \tag{83.86}$$

The mth ray intersects both cell surfaces, inducing a mesh $\{s_{lm}\}$, and the spherical surfaces through cell centers, inducing a second mesh $s_{l+(1/2),m}$, where

$$s_{lm} = (r_l^2 - p_m^2)^{1/2}, \tag{83.87a}$$

and

$$s_{l+(1/2),m} = [r^2_{l+(1/2)} - p_m^2]^{1/2}. \tag{83.87b}$$

For rays inside the core $(p_m \le p_C)$, $\{s_{lm}\}$ is defined for $l = 1, \ldots, I + 1$, and $\{s_{l+(1/2),m}\}$ for $l = 1, \ldots, I$. For rays outside the core, $\{s_{lm}\}$ is defined for $l = m - C + 1, \ldots, I + 1$, and $\{s_{l+(1/2),m}\}$ for $l = m - C, \ldots, I$. By symmetry, we need to calculate the solution only to the right of the vertical axis in Figure 83.2, and to the right of the core.

We write a difference-equation representation of (83.85) in terms of $j_{i+(1/2),mn} \equiv j[s_{i+(1/2),m}, p_m, \nu_n]$, similar to (83.31) with $\mu \equiv 1$. We define

$$\Delta\tau_{imn} \equiv \chi_{i-(1/2),n}[s_{im} - s_{i-(1/2),m}] + \chi_{i+(1/2),n}[s_{i+(1/2),m} - s_{im}] \tag{83.88a}$$

and

$$\Delta\tau_{i+(1/2),mn} \equiv \tfrac{1}{2}(\Delta\tau_{imn} + \Delta\tau_{i+1,mn}). \tag{83.88b}$$

For rays outside the core $(m > C)$, we can apply (83.85) at $l = m - C, \ldots, I$. At the axis of symmetry we require that the fictitious quantity $j_{[m-C-(1/2)],mn} \equiv j_{[m-C+(1/2)],mn}$ to obtain an equation containing only $j_{[m-C+(1/2)],mn}$ and $j_{[m-C+(3/2)],mn}$. At the outer boundary we manipulate difference representations of (83.83) and (83.84), using $j(R) \equiv h(R)$ (no incident radiation), as in (83.32) to (83.36) to obtain an equation containing only $j_{I+(1/2),mn}$ and $j_{I-(1/2),mn}$. For rays that intersect the core, we obtain an inner boundary condition by imposing a known flux from the core [see (**M2,** 252–254) for further details].

Along each ray we have a tridiagonal system of order L, where L ranges from 1 to I. Summing over all rays, for all frequencies, we find the total computing effort scales as $c'I^2N$ (assuming $I \gg C$). On vector computers, the equations for all frequencies on a given ray can be solved in parallel.

An alternative way of doing the formal solution in spherical geometry is to develop a finite difference representation of (76.9) with r and μ as independent variables. Suppose we start with the transfer equation in conservation form:

$$3\mu \frac{\partial(r^2 I_\nu)}{\partial(r^3)} + \frac{1}{r}\frac{\partial}{\partial\mu}[(1-\mu^2)I_\nu] = \eta_\nu - \chi_\nu I_\nu. \tag{83.89}$$

Writing (83.89) for $\pm\mu$, and taking the sum and difference of these equations, we have

$$3\mu \frac{\partial(r^2 h_\nu)}{\partial(r^3)} + \frac{1}{r}\frac{\partial}{\partial\mu}[(1-\mu^2)h_\nu] = \eta_\nu - \chi_\nu j_\nu \tag{83.90}$$

and

$$3\mu \frac{\partial(r^2 j_\nu)}{\partial(r^3)} + \frac{1}{r}\frac{\partial}{\partial\mu}[(1-\mu^2)j_\nu] = -\chi_\nu h_\nu, \tag{83.91}$$

where j_ν and h_ν are defined as in (83.12) and (83.13), for $0 \leq \mu \leq 1$.

We discretize (83.90) and (83.91) on a radial mesh with h_ν located at cell surfaces $\{r_i\}$, $(i = 1, \ldots, I+1)$, and j_ν at cell centers $\{r_{i+(1/2)}\}$, $(i = 1, \ldots, I)$, and introduce both a frequency mesh $\{\nu_n\}$ and angular mesh $\{\mu_m\}$; a typical choice for $\{\mu_m\}$ is a double-Gauss quadrature of order M. Equations (83.90) and (83.91) then become

$$\begin{aligned} 3\mu_m(r_{i+1}^2 h_{i+1,mn} - r_i^2 h_{imn})/(r_{i+1}^3 - r_i^3) + [2r_{i+(1/2)}]^{-1} \sum_{m'} D_{mm'}(h_{im'n} + h_{i+1,m'n}) \\ = \eta_{i+(1/2),n} - \chi_{i+(1/2),n} j_{i+(1/2),mn}, \qquad (m = 1, \ldots, M), \end{aligned} \tag{83.92}$$

and

$$\begin{aligned} 3\mu_m[r_{i+(1/2)}^2 j_{i+(1/2),mn} - r_{i-(1/2),mn}^2 j_{i-(1/2),mn}]/[r_{i+(1/2)}^3 - r_{i-(1/2)}^3] \\ + (2r_i)^{-1} \sum_{m'} D_{mm'}[j_{i-(1/2),m'n} + j_{i+(1/2),m'n}] \\ = -\chi_{in} h_{imn}, \qquad (m = 1, \ldots, M), \end{aligned} \tag{83.93}$$

where

$$\chi_{in} \equiv \{\chi_{i-(1/2)}[r_i - r_{i-(1/2)}] + \chi_{i+(1/2)}[r_{i+(1/2)} - r_i]\}/[r_{i+(1/2)} - r_{i-(1/2)}]. \tag{83.94}$$

The matrix $D_{mm'}$ is a discrete representation of the angle-derivative terms in (83.90) and (83.91), evaluated at μ_m; it may be constructed in several different ways. One approach (**C1**), (**C2**), (**C6,** §91), (**N1**) is to assume that $j(\mu)$ and $h(\mu)$ are represented by (unique) interpolating polynomials of order $(M-1)$ determined by the angle mesh $\{\mu_m\}$. These polynomials can be written conveniently as

$$j(\mu) \approx \sum_{i=1}^{M} L_i(\mu) j(\mu_i), \tag{83.95}$$

and similarly for $h(\mu)$. Here $L_i(\mu)$ is the Lagrange polynomial

$$L_i(\mu) \equiv \frac{(\mu-\mu_1)\ldots(\mu-\mu_{i-1})(\mu-\mu_{i+1})\ldots(\mu-\mu_M)}{(\mu_i-\mu_1)\ldots(\mu_i-\mu_{i-1})(\mu_i-\mu_{i+1})\ldots(\mu_i-\mu_M)} \tag{83.96}$$

which is defined such that $L_i(\mu_j) = \delta_{ij}$. Then

$$(\partial j/\partial \mu)_m = \sum_{i=1}^{M} (dL_i/d\mu)_{\mu_m} j(\mu_i) \tag{83.97}$$

whence

$$D_{mm'} = (1-\mu_m^2)(dL_{m'}/d\mu)_{\mu_m} - 2\mu_m\,\delta_{mm'}. \tag{83.98}$$

The derivatives $(dL_i/d\mu)$ are easily calculated analytically from (83.96). A related approach is to represent $(1-\mu^2)j(\mu)$ and $(1-\mu^2)h(\mu)$ instead of $j(\mu)$ and $h(\mu)$ by the interpolating polynomials.

The **D** matrix given by (83.98) is full, and for large M (needed for accuracy) has large elements of alternating sign. This is a characteristic of all high-order differentiation formulae, reflecting the fact that high-order polynomials can oscillate wildly over their interval of definition; numerical noise or other errors in the solution then tend to be amplified, and the computation may become unstable.

The difficulty just described can be circumvented by using a *discrete-space method* similar to Carlson's *S_N method* (**C4**), (**L4**), (**L5**). Here, one assumes that the discrete ordinates $j(\mu_m)$ and $h(\mu_m)$ represent j and h within an angular cell $[\mu_{m-(1/2)}, \mu_{m+(1/2)}]$. The cell boundaries are taken to be $\mu_{1/2} \equiv 0$, and

$$\mu_{m+(1/2)} \equiv \mu_{m-(1/2)} + b_m, \qquad (m = 1, \ldots, M), \tag{83.99}$$

where b_m is the quadrature weight associated with μ_m. The properties of quadrature formulae assure that $\mu_{m-(1/2)} < \mu_m < \mu_{m+(1/2)}$, and that $\mu_{M+(1/2)} \equiv 1$.

Thus, integrating (83.90) over $[\mu_{m-(1/2)}, \mu_{m+(1/2)}]$, and suppressing depth and frequency subscripts, we obtain

$$\begin{aligned} 3b_m\mu_m \frac{\partial(r^2 h_m)}{\partial(r^3)} + \frac{1}{r}\{[1-\mu_{m+(1/2)}^2]h_{m+(1/2)} - [1-\mu_{m-(1/2)}^2]h_{m-(1/2)}\} \\ = b_m(\eta - \chi j_m), \qquad (m = 1, \ldots, M). \end{aligned} \tag{83.100}$$

Noticing that by symmetry $h_{1/2} = h(\mu = 0) \equiv 0$, we see that (83.100) has the desirable property of being strictly conservative: when summed over all m, it yields the zeroth moment equation (83.69) exactly.

We must now represent $h_{n\pm(1/2)}$ in terms of the discrete ordinates $\{h_m\}$; we then discretize the spatial operator as before, and obtain an equation of the form (83.92). To maximize stability of the solution, one may represent $h(\mu)$ by a linear spline (**P2**), so that

$$h_{m+(1/2)} = \{[\mu_{m+1} - \mu_{m+(1/2)}]h_m + [\mu_{m+(1/2)} - \mu_m]h_{m+1}\}/(\mu_{m+1} - \mu_m), \qquad (m=1,\dots,M-1). \tag{83.101}$$

We need not specify $h_{M+(1/2)}$ as its coefficient in (83.100) vanishes because $\mu_{M+(1/2)} \equiv 1$. Substituting (83.101) into (83.100), and recalling $h_{1/2} \equiv 0$, we find that $\mathbf{D}$ is a tridiagonal matrix with elements

$$b_m D_{m,m-1} = -[1-\mu^2_{m-(1/2)}][\mu_m - \mu_{m-(1/2)}]/(\mu_m - \mu_{m-1}), \qquad (m=2,\dots,M), \tag{83.102a}$$

$$b_m D_{mm} = \frac{[1-\mu^2_{m+(1/2)}][\mu_{m+1} - \mu_{m+(1/2)}]}{(\mu_{m+1} - \mu_m)} - \frac{[1-\mu^2_{m-(1/2)}][\mu_{m-(1/2)} - \mu_{m-1}]}{(\mu_m - \mu_{m-1})}, \qquad (m=2,\dots,M), \tag{83.102b}$$

$$b_m D_{m,m+1} = [1-\mu^2_{m+(1/2)}][\mu_{m+(1/2)} - \mu_m]/(\mu_{m+1} - \mu_m), \qquad (m=1,\dots,M-1), \tag{83.102c}$$

and

$$b_1 D_{11} = (1-\mu^2_{3/2})(\mu_2 - \mu_{3/2})/(\mu_2 - \mu_1). \tag{83.102d}$$

In developing a discrete-space representation of (83.91) we first rewrite it as

$$3\mu^2 \frac{\partial(r^2 j_\nu)}{\partial(r^3)} + \frac{(\mu^2-1)}{r} j_\nu + \frac{1}{r}\frac{\partial}{\partial\mu}[\mu(1-\mu^2)j_\nu] = -\chi_\nu \mu h_\nu, \tag{83.103}$$

which when integrated over $[\mu_{m-(1/2)}, \mu_{m+(1/2)}]$ becomes

$$3b_m\mu_m^2 \frac{\partial(r^2 j_m)}{\partial r^3} + b_m \frac{(\mu_m^2-1)}{r} j_m + \frac{1}{r}\{\mu_{m+(1/2)}[1-\mu^2_{m+(1/2)}]j_{m+(1/2)} - \mu_{m-(1/2)}[1-\mu^2_{m-(1/2)}]j_{m-(1/2)}\} = -\chi b_m \mu_m h_m, \qquad (m=1,\dots,M). \tag{83.104}$$

Recalling that $\mu_{1/2} \equiv 0$ and $\mu_{M+(1/2)} \equiv 1$, we see that (83.104) has the desirable property that when summed over m it yields the first-moment equation (83.70) exactly. Discretizing (83.104) we obtain

$$3\mu_m^2[r^2_{i+(1/2)} j_{i+(1/2),mn} - r^2_{i-(1/2)} j_{i-(1/2),mn}]/[r^3_{i+(1/2)} - r^3_{i-(1/2)}] + (2r_i)^{-1}\sum_{m'} D'_{mm'}[j_{i-(1/2),mn} + j_{i+(1/2),mn}] = -\chi_{in}\mu_m h_{imn}, \qquad (m=1,\dots,M). \tag{83.105}$$

Representing $j(\mu)$ by a linear spline as in (83.101), one finds that $\mathbf{D}'$ is tridiagonal with elements

$$b_m D'_{m,m-1} = -\mu_{m-(1/2)}[1-\mu^2_{m-(1/2)}][\mu_m - \mu_{m-(1/2)}]/(\mu_m - \mu_{m-1}), \qquad (m=2,\ldots,M), \qquad (83.106a)$$

$$b_m D'_{mm} = (\mu_m^2 - 1) + \frac{\mu_{m+(1/2)}[1-\mu^2_{m+(1/2)}][\mu_{m+1} - \mu_{m+(1/2)}]}{(\mu_{m+1} - \mu_m)} - \frac{\mu_{m-(1/2)}[1-\mu^2_{m-(1/2)}][\mu_{m-(1/2)} - \mu_{m-1}]}{(\mu_m - \mu_{m-1})} \qquad (m=1,\ldots,M), \qquad (83.106b)$$

and

$$b_m D'_{m,m+1} = \mu_{m+(1/2)}[1-\mu^2_{m+(1/2)}][\mu_{m+(1/2)} - \mu_m]/(\mu_{m+1} - \mu_m), \qquad (m=1,\ldots,M-1).$$

If the discrete-space equations are formulated in terms of the intensity $I^{\pm}$, instead of j and h, the fact that a photon trajectory always has a larger value of μ at the point of absorption than at the point of creation argues (**L6**) that the angle derivative should be represented by an upstream difference, that is, $I_{m+(1/2)} \equiv I_m$. This argument does not apply, however, to j and h, which mix information at $\pm\mu$.

Equations (83.92) and (83.93), with suitable boundary conditions, pose a coupled system of the general form

$$-\mathbf{A}_i \mathbf{j}_{i+(1/2)} + \mathbf{B}_i \mathbf{h}_i - \mathbf{C}_i \mathbf{j}_{i-(1/2)} = \mathbf{D}_i, \qquad (i = I+1, \ldots, 1), \qquad (83.107)$$

and

$$-\mathbf{E}_{i+(1/2)} \mathbf{h}_{i+1} + \mathbf{F}_{i+(1/2)} \mathbf{j}_{i+(1/2)} - \mathbf{G}_{i+(1/2)} \mathbf{h}_i = \mathbf{H}_{i+(1/2)}, \qquad (i = I, \ldots, 1). \qquad (83.108)$$

Here $\mathbf{j}_{i+(1/2)}$ and $\mathbf{h}_i$ are vectors of length M containing the angle components $\mathbf{j}_{i+(1/2),mn}$ and $\mathbf{h}_{imn}$; the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{E}$, $\mathbf{F}$, $\mathbf{G}$ are all of dimension $M \times M$. The boundary conditions imply that $\mathbf{A}_{I+1} \equiv 0$ and $\mathbf{C}_1 \equiv 0$. Because the equations are already angle coupled, scattering terms in the source function can be written out explicitly as in (83.26) at no extra computational cost. Equations (83.107) and (83.108) are solved by Gaussian elimination, starting with the forward elimination

$$\mathbf{h}_i = \mathbf{K}_i \mathbf{j}_{i-(1/2)} + \mathbf{L}_i \qquad (83.109)$$

and

$$\mathbf{j}_{i-(1/2)} = \mathbf{M}_{i-(1/2)} \mathbf{h}_{i-1} + \mathbf{N}_{i-(1/2)}, \qquad (83.110)$$

for $i = I+1, \ldots, 2$, where

$$\mathbf{K}_i \equiv [\mathbf{B}_i - \mathbf{A}_i \mathbf{M}_{i+(1/2)}]^{-1} \mathbf{C}_i, \qquad (83.111)$$

$$\mathbf{L}_i \equiv [\mathbf{B}_i - \mathbf{A}_i \mathbf{M}_{i+(1/2)}]^{-1} [\mathbf{D}_i + \mathbf{A}_i \mathbf{N}_{i+(1/2)}], \qquad (83.112)$$

$$\mathbf{M}_{i-(1/2)} \equiv [\mathbf{F}_{i-(1/2)} - \mathbf{E}_{i-(1/2)} \mathbf{K}_i]^{-1} \mathbf{G}_{i-(1/2)}, \qquad (83.113)$$

and

$$\mathbf{N}_{i-(1/2)} \equiv [\mathbf{F}_{i-(1/2)} - \mathbf{E}_{i-(1/2)} \mathbf{K}_i]^{-1} [\mathbf{H}_{i-(1/2)} + \mathbf{E}_{i-(1/2)} \mathbf{L}_i]. \qquad (83.114)$$

The forward sweep is completed by applying (83.109) at $i=1$, where $\mathbf{C}_1 \equiv 0$ implies $\mathbf{h}_1 \equiv \mathbf{L}_1$. We then back-substitute into (83.110) and (83.109) to determine $\mathbf{j}_{i+(1/2)}$ and $\mathbf{h}_{i+1}$ for $i=1, \ldots, I$. The computational effort for I radial shells, M angles, and N frequencies scales as cIM^3N.

Both the tangent-ray and discrete-space schemes have advantages and disadvantages. The tangent-ray method is expensive for large I because of the large number of tangent rays and mesh points per ray. However the solution along each ray is a true formal solution and is relatively cheap. Furthermore, this method provides good angular resolution of the radiation field (particularly when $R/r_c \gg 1$ and the radiation field becomes strongly peaked towards $\mu=1$) because it samples all source shells. The discrete-space scheme uses M fixed angles and is economical when M is small. But because it is no longer a true formal solution (the equations are angle coupled) the computational expense rises rapidly with M. For small M the angular resolution may be inadequate when $R/r_c \gg 1$ and/or the radiation field is strongly forward peaked. But the method may be completely adequate, with small M, for problems in which a geometrically thin transport layer (e.g., a stellar atmosphere) surrounds a large diffusion region (e.g., a stellar interior) within which sphericity effects are important [because (r/r_c) varies over a large range] but the radiation field is nearly isotropic.

Having found j_ν at all depths and frequencies by either method, we can evaluate the f_ν's and q_ν's needed in the moment equations. We iterate between the moment equations and the formal solution as in the planar case. If J iterations are required for convergence, the total computing effort scales as $J(cIN^3+c'I^2N)$ or $J(cIN^3+c'IM^3N)$ for the tangent-ray and discrete-space methods respectively.

6.6 Statistical Equilibrium in the Presence of a Radiation Field

The occupation numbers of atomic levels in a radiating fluid are not, in general, those predicted by equilibrium statistical mechanics for local values of the temperature and density (LTE). In some cases the departures from LTE are severe; generally they are driven by radiative processes that deviate from equilibrium values in regions where photons escape efficiently through a boundary. Using a stellar atmosphere as an example, we first examine the microscopic requirements of LTE, and show that they are unfulfilled. We then formulate rate equations that determine the state of the material from local values of the temperature and density and the radiation field.

84. The Microscopic Implications of LTE

The hypothesis of LTE makes several tacit assumptions; let us examine some of these critically.

DETAILED BALANCE

In thermodynamic equilibrium, *all* processes are in detailed balance, that is, every transition, of any kind, is exactly canceled by its inverse. Detailed balance holds for collisional processes so long as the particle distribution function is Maxwellian; collisions then occur at their equilibrium rates (per particle) at the local kinetic temperature. If these were the only processes operating, we would have LTE.

However, radiative processes compete with collisions. When they occur at their equilibrium rates (e.g., the radiation field is Planckian), they also are in detailed balance, and help drive the material toward LTE. But if the radiation field has a nonequilibrium distribution the radiative rates can be out of balance and will tend to drive the material away from LTE. In the interior (or deep in the atmosphere) of a star the radiation field does, in fact, thermalize to the Planck function, and LTE obtains. However, at the surface of a star the radiation field is out of equilibrium in two important respects. First, there are no incoming photons, hence the field is anisotropic, and is *dilute* because the mean intensity (which sets absorption rates) averages over a hemisphere containing no radiation. More important, the outward-moving photons at the surface originate mainly from layers at unit optical depth in the atmosphere; these layers in general have physical properties (e.g., temperature) substantially different from the surface layers. Hence even for Planckian emission from the deeper layers, at the surface the characteristic radiation temperature T_R can be quite different from the local kinetic temperature T_k. The effect is largest at frequencies where the material is most transparent and the unit optical-depth surface lies deepest. Moreover, when $h\nu/kT \gg 1$, $B_\nu(T)$ varies as $e^{-h\nu/kT}$, hence $B_\nu(T_R)$ may differ from $B_\nu(T_k)$ by orders of magnitude even for a modest difference between T_R and T_k. Thus the frequency spectrum of the intensity near the surface will generally be strongly non-Planckian, and radiative rates will be far from detailed balance.

We note in passing that even when the material is not in LTE we can use detailed-balancing arguments to determine *rate coefficients*; for example, we can express collisional de-excitation rates in terms of excitation rates. We shall use this device repeatedly.

THE PARTICLE VELOCITY-DISTRIBUTION FUNCTION

In equilibrium, all particles have Maxwellian velocity distributions at a single kinetic temperature; this distribution is the unique result of elastic collisions among particles (cf. §§8 and 9). However, in general both radiative and inelastic collisional processes can perturb the equilibrium; for example, a collisional excitation of an atom by an electron lowers the electron's energy by a discrete amount, while a recombination prevents further elastic collisions. Thus establishment of a Maxwellian distribution hinges on whether the elastic collisions occur much more frequently than inelastic or radiative processes.

Characteristic times for these competing processes are the self-collision time t_c for electrons [cf. (10.26)], the radiative recombination time t_{rr}, and the inelastic collision time t_{ic}. For conditions in stellar atmospheres, conservative estimates show that $t_c/t_{rr} \sim 10^{-7}$ and $t_c/t_{ic} \lesssim 10^{-3}$, so the elastic collisions always dominate, and the velocity distribution should be Maxwellian (**B2**), (**M2,** §5–3). Moreover, demanding steady state in an atmosphere of hydrogen atoms, protons, electrons, and radiation while allowing energy exchange among all components, one finds (**B1**) that $|T_{\text{ion}} - T_{\text{elec}}| \lesssim 10^{-3} T_{\text{elec}}$ provided that the electron density $n_e \gtrsim 10^{10}$. Thus in what follows we can safely assume that all particles are, in fact, Maxwellian at a single temperature.

EXCITATION AND IONIZATION EQUILIBRIUM

When collisional excitation and ionization rates (and their inverses) dominate the corresponding radiative rates, LTE should prevail; this is the case in dense laboratory plasmas and stellar interiors, but not in a stellar atmosphere. One can show (**B2**) that in the Sun ($T \sim 6000$ K) the ratio of radiative to collisional ionization rates ranges from about 2 to about 10^3 for atoms with ionization potentials $\varepsilon_{\text{ion}} \sim 1$ eV and ~ 8 eV respectively. For an *O*-star ($T \sim 3 \times 10^4$ K) these ratios are 0.2 and 20; but atoms with $\varepsilon_{\text{ion}} \sim 1$ eV are no longer important because they are completely ionized. Hence photoionization rates generally vastly exceed collisional ionization rates; similarly, radiative recombinations outweigh collisional recombinations.

Similarly, radiative excitation rates in the Sun are from 6 to 300 times larger than collision rates for transitions in the spectral range $3000\ \text{Å} \lesssim \lambda \lesssim 9000\ \text{Å}$. For an *O*-star the radiative rates exceed collisional only for $\lambda \lesssim 5000$ Å; but the bulk of the radiation emerges in the far ultraviolet for such stars, thus for all practical purposes radiative processes far outweigh collisions.

We thus see that in stellar atmospheres both excitation and ionization equilibria can be driven away from LTE by radiative processes. We next must ask "over what depth-range can radiative rates depart markedly from their equilibrium values?". Early discussions [e.g., (**U1**), (**B2**)] concluded that equilibrium radiative rates in any transition are attained at optical depths greater than unity in that transition. However, this conclusion is false [see (**T2,** 141–147) or (**M2,** §5–3)] because of the effects of scattering, which allow J_ν to depart from B_ν (hence departures from LTE to occur) to a depth equal to a photon destruction length (cf. §83). Furthermore, because level populations are determined by *all* rates within the transition array, departure of even one transition from equilibrium tends to drive the population of *all* levels away from LTE. Only at depths where the radiation field is thermalized at *all* frequencies (i.e., $\tau_\nu \gtrsim \xi_\nu^{-1/2}$ for all ν) is the recovery of LTE guaranteed.

In summary, in a stellar atmosphere (whence the photons we observe originate!) one must, in general, allow for departures from LTE. The machinery to do so is provided by *non-LTE rate equations*, which we now consider.

85. Non-LTE Rate Equations

GENERAL FORM

We derive the rate equations for the general case of a moving medium. Let n_{ik} be the number density of particles in level i of chemical species k in a fixed volume. The rate of change of n_{ik} is determined by the net rate of flow of particles into the volume and the net rate of transitions into level i from all other levels j by atomic processes. Thus

$$(\partial n_{ik}/\partial t) = -\nabla\cdot(n_{ik}\mathbf{v}) + \sum_{j\neq i} n_{jk}P_{ji}^k - n_{ik}\sum_{j\neq i} P_{ij}^k; \tag{85.1}$$

here P_{ij}^k is the total (radiative plus collisional) transition rate from level i to level j. If we sum (85.1) over all states of species k, all atomic rates cancel term by term, and, writing $N_k \equiv \sum_i n_{ik}$, we obtain a continuity equation for species k:

$$(\partial N_k/\partial t) = -\nabla\cdot(N_k\mathbf{v}). \tag{85.2}$$

Multiplying (85.2) by m_k, the mass of species k, summing over species, and noting that $\rho = \sum_k m_k N_k$, we recover the continuity equation

$$(\partial\rho/\partial t) = -\nabla\cdot(\rho\mathbf{v}). \tag{85.3}$$

Using (85.3) to eliminate $\nabla\cdot\mathbf{v}$ we can rewrite (85.1) in the Lagrangean form

$$(Dn_{ik}/Dt) - (n_{ik}/\rho)(D\rho/Dt) = \sum_{j\neq i} n_{jk}P_{ji}^k - n_{ik}\sum_{j\neq i} P_{ij}^k, \tag{85.4}$$

or, more instructively, as

$$\rho[D(n_{ik}/\rho)Dt] = \sum_{j\neq i} n_{jk}P_{ji}^k - n_{ik}\sum_{j\neq i} P_{ij}^k. \tag{85.5}$$

Equation (85.5) states that in a material element, the rate of change of the number of particles, per unit mass, in a particular level equals the net number, per unit mass, entering that level via atomic transitions. The Lagrangean form of (85.2) is

$$D(N_k/\rho)/Dt = 0, \tag{85.6}$$

which states that the total number of particles of a chemical species, per unit mass, in a material element is constant.

For steady flow (85.1) simplifies to

$$\sum_{j\neq i} n_{jk}P_{ji}^k - n_{ik}\sum_{j\neq i} P_{ij}^k - \nabla\cdot(n_{ik}\mathbf{v}) = 0, \tag{85.7}$$

and for a static medium it becomes

$$\sum_{j\neq i} n_{jk}P^k_{ji} - n_{ik}\sum_{j\neq i} P^k_{ij} = 0. \tag{85.8}$$

In what follows we concentrate mainly on formulating and solving (85.8), which illustrates all the essential physics; we generalize to time-dependent flows at the end of the section. In all cases, all rates are to be evaluated in the comoving fluid frame.

COLLISION RATES

In an ionized plasma, atomic collisions with charged particles dominate all others because of the long range of Coulomb interactions. Furthermore, because the collision rate is proportional to the flux (hence velocity) of the incident particles, we usually need consider only electrons, whose velocities are a factor of $(Am_{\rm H}/m_e)^{1/2} \approx 43A^{1/2}$ greater than those of ions of atomic weight A.

Let $\sigma_{ij}(v)$ be the cross section for transitions $(i \rightarrow j)$ produced by collisions with electrons moving at speed v. The number of collisional excitations $(i \rightarrow j)$ is

$$n_i C_{ij} = n_i n_e \int_{v_0}^{\infty} \sigma_{ij}(v) f(v) v\, dv, \tag{85.9}$$

where $f(v)$ is the Maxwellian velocity distribution and v_0 is the speed corresponding to the threshold energy E_{ij} of the transition (i.e., $\frac{1}{2}m_e v_0^2 = E_{ij}$). In equilibrium we have $n_i^* C_{ij} = n_j^* C_{ji}$, which allows us to compute the collisional deexcitation rate $(j \rightarrow i)$ as

$$n_j C_{ji} = n_j (n_i/n_j)^* C_{ij} = n_j (g_i e^{h\nu_{ij}/kT}/g_j) C_{ij}. \tag{85.10}$$

Similarly, the collisional ionization rate from level i is

$$n_i C_{i\kappa} = n_i n_e \int_{v_0}^{\infty} \sigma_{i\kappa}(v) f(v) v\, dv, \tag{85.11}$$

and the collisional recombination rate is

$$n_\kappa C_{\kappa i} = n_\kappa (n_i/n_\kappa)^* C_{i\kappa} = n_i^* C_{i\kappa} \tag{85.12}$$

where n_i^* is computed using actual electron and ion densities, that is, $(n_i/n_\kappa)^* \equiv n_e \Phi_{i\kappa}(T)$.

RADIATIVE RATES

The number of upward radiative bound-bound transitions $(i \rightarrow j)$ is

$$\begin{aligned} n_i R_{ij} &= n_i B_{ij} \int \phi_\nu J_\nu \, d\nu \equiv n_i B_{ij} \bar{J}_{ij} \\ &= n_i (4\pi\alpha_{ij}/h\nu_{ij}) \bar{J}_{ij} = n_i 4\pi \int (h\nu)^{-1} \alpha_{ij}(\nu) J_\nu \, d\nu. \end{aligned} \tag{85.13}$$

The last form will prove useful later. We emphasize that (85.13) applies when all quantities are measured in the comoving frame. In the laboratory frame $\bar{J}$ is the much more complicated expression

$$\bar{J}_{ij} = (4\pi)^{-1} \int d\nu \oint d\omega \phi[\nu(1-\mathbf{n}\cdot\mathbf{v}/c)] I(\mathbf{n}, \nu). \tag{85.14}$$

The total number (spontaneous plus induced) of downward radiative transitions $(j \rightarrow i)$ is

$$n_j R'_{ji} = n_j(A_{ji} + B_{ji}\bar{J}_{ij}) = n_j(4\pi\alpha_{ij}/h\nu_{ij})(g_i/g_j)[(2h\nu_{ij}^3/c^2) + \bar{J}_{ij}]. \tag{85.15}$$

A prime has been added to R'_{ji} to reserve the unadorned symbol for a different use below. We cast (85.15) into a more useful form by factoring out the quantity $(n_i/n_j)^* = (g_i/g_j)e^{h\nu_{ij}/kT}$ from the right-hand side to obtain

$$n_j R'_{ji} \equiv n_j\left(\frac{n_i}{n_j}\right)^* R_{ji} = n_j\left(\frac{n_i}{n_j}\right)^*\left[4\pi \int \frac{\alpha_{ij}(\nu)}{h\nu}\left(\frac{2h\nu^3}{c^2} + J_\nu\right)e^{-h\nu/kT}\, d\nu\right]. \tag{85.16}$$

In (85.16) we can take ν-dependent factors inside the integral because the line profile ϕ_ν varies swiftly. Despite its apparently cumbersome form, this expression will allow us to systematize notation effectively.

Now consider bound-free transitions. The photoionization rate is

$$n_i R_{i\kappa} = n_i 4\pi \int_{\nu_0}^{\infty} [\alpha_{i\kappa}(\nu) J_\nu / h\nu]\, d\nu. \tag{85.17}$$

To calculate the spontaneous recombination rate we invoke a detailed balancing argument. In TE the number of spontaneous recombinations equals the number of photoionizations corrected for stimulated emissions, that is,

$$(n_\kappa R'_{\kappa i})^*_{\text{spon}} = n_i^* 4\pi \int_{\nu_0}^{\infty} [\alpha_{i\kappa}(\nu)/h\nu] B_\nu (1 - e^{-h\nu/kT})\, d\nu. \tag{85.18}$$

The corresponding nonequilibrium rate is obtained by calculating n_i^* using the actual ion density n_κ; thus

$$\begin{aligned}(n_\kappa R'_{\kappa i})_{\text{spon}} &= n_\kappa (n_i/n_\kappa)^* 4\pi \int_{\nu_0}^{\infty} [\alpha_{i\kappa}(\nu)/h\nu] B_\nu (1 - e^{-h\nu/kT})\, d\nu \\ &= n_\kappa (n_i/n_\kappa)^* 4\pi \int_{\nu_0}^{\infty} [\alpha_{i\kappa}(\nu)/h\nu](2h\nu^3/c^2) e^{-h\nu/kT}\, d\nu.\end{aligned} \tag{85.19}$$

The equilibrium number of stimulated emissions is the term containing $e^{-h\nu/kT}$ in (85.18); to obtain the nonequilibrium number we replace B_ν by J_ν, and calculate n_i^* using the actual ion density. Thus

$$(n_\kappa R'_{\kappa i})_{\text{stim}} = n_\kappa (n_i/n_\kappa)^* 4\pi \int_{\nu_0}^{\infty} [\alpha_{i\kappa}(\nu)/h\nu] J_\nu e^{-h\nu/kT}\, d\nu. \tag{85.20}$$

Combining (85.19) and (85.20) we get the total recombination rate

$$n_\kappa R'_{\kappa i} \equiv n_\kappa (n_i/n_\kappa)^* R_{\kappa i} = n_\kappa \left(\frac{n_i}{n_\kappa}\right)^* 4\pi \int_{\nu_0}^{\infty} \frac{\alpha_{i\kappa}(\nu)}{h\nu} \left(\frac{2h\nu^3}{c^2} + J_\nu\right) e^{-h\nu/kT}\, d\nu. \tag{85.21}$$

Comparing (85.13) with (85.17), and (85.16) with (85.21), shows that the notation is systematized by writing all upward radiative rates ($i \to j$), whether j is bound or free, as $n_i R_{ij}$ where

$$R_{ij} \equiv 4\pi \int_{\nu_0}^{\infty} [\alpha_{ij}(\nu) J_\nu / h\nu]\, d\nu, \tag{85.22}$$

and all downward radiative rates ($j \to i$) as $n_j (n_i/n_j)^* R_{ji}$, where

$$R_{ji} \equiv 4\pi \int_{\nu_0}^{\infty} [\alpha_{ij}(\nu)/h\nu][(2h\nu^3/c^2) + J_\nu] e^{-h\nu/kT}\, d\nu. \tag{85.23}$$

Notice that in equilibrium $R_{ij}^* \equiv R_{ji}^*$.

SOLUTION OF THE RATE EQUATIONS

We now assemble individual rates into equations from which occupation numbers can be computed. For the present we take as given the temperature T, the total particle density N, and the mean intensity J_ν at all frequencies. One can regard the temperature and gas pressure ($p = NkT$) as coming from a solution of the momentum and energy equations, and J_ν from a solution of the transfer equation. For simplicity we consider a gas of pure hydrogen.

For each bound level we have a *rate equation* of the form

$$\begin{aligned} \sum_{j<i} n_j (R_{ji} + C_{ji}) - n_i \left[\sum_{j<i} (n_j/n_i)^* (R_{ij} + C_{ji}) + \sum_{j>i}^{\kappa} (R_{ij} + C_{ij}) \right] & \\ + \sum_{j>i}^{\kappa} n_j (n_i/n_j)^* (R_{ji} + C_{ij}) = 0, \qquad (i = 1, \ldots, L), & \end{aligned} \tag{85.24}$$

where L is the number of bound levels. We can also write a total ionization equation

$$\sum_{i=1}^{L} n_i (R_{i\kappa} + C_{i\kappa}) - n_p \sum_{i=1}^{L} (n_i/n_p)^* (R_{\kappa i} + C_{i\kappa}) = 0, \tag{85.25}$$

where n_p is the proton density. However (85.25) contains no new information because it is merely the sum of (85.24) over all bound levels.

We thus have L equations in $L+2$ variables: $n_1, \ldots, n_L, n_p, n_e$. To close the system we invoke *charge conservation*,

$$n_p = n_e, \tag{85.26}$$

and *number conservation*,

$$\sum_{i=1}^{L} n_i + n_p + n_e = N. \tag{85.27}$$

Clearly (85.26) is trivial and could be used to eliminate n_e or n_p; we retain it because in a gas mixture several species may contribute electrons.

The system composed of (85.24) and (85.26) can be written

$$\mathbf{A}\mathbf{n} = \mathbf{B}, \tag{85.28}$$

where $\mathbf{n}$ is the column vector

$$\mathbf{n} = (n_1, \ldots, n_L, n_p), \tag{85.29}$$

$\mathbf{A}$ is a matrix of dimension $(L+1)\times(L+1)$ containing the transition rates, and $\mathbf{B}$ is the column vector

$$\mathbf{B} = (0, 0, \ldots, 0, n_e). \tag{85.30}$$

If T, n_e, and J_ν are given, all elements of $\mathbf{A}$ and $\mathbf{B}$ are known, and we can solve the linear system (85.28) for $\mathbf{n}$ by standard numerical methods. But if we are given N, not n_e, the unknown n_e must also be determined from the complete system comprising (85.28) and (85.27); this system is *nonlinear* because n_e appears in collision and recombination rates in elements of $\mathbf{A}$, which in turn multiply the level populations $n_1, \ldots, n_p$. The nonlinear system is solved by an iterative linearization scheme.

Suppose we have an estimate n_e^0 of the electron density (say from assuming LTE), which we use to calculate estimates $\mathbf{A}^0$ and $\mathbf{B}^0$ of $\mathbf{A}$ and $\mathbf{B}$, and we solve $\mathbf{A}^0\mathbf{n}^0 = \mathbf{B}^0$ for an approximate set of occupation numbers $\mathbf{n}^0$. In general, n_e^0 and $\mathbf{n}^0$ will fail to satisfy (85.27), and we must improve these initial estimates. To do so, we set $\mathbf{n} = \mathbf{n}^0 + \delta\mathbf{n}$ and $n_e = n_e^0 + \delta n_e$, and expand (85.28) to first order, obtaining

$$\mathbf{A}^0(\mathbf{n}^0 + \delta\mathbf{n}) + \mathbf{n}^0 \cdot (\partial\mathbf{A}/\partial n_e)^0\, \delta n_e = \mathbf{B}^0 + (\partial\mathbf{B}/\partial n_e)^0\, \delta n_e. \tag{85.31}$$

Here

$$[\mathbf{n}^0 \cdot (\partial\mathbf{A}/\partial n_e)^0]_i = \sum_j (\partial A_{ij}/\partial n_e)^0 n_j^0, \tag{85.32}$$

and

$$(\partial\mathbf{B}/\partial n_e) = (0, \ldots, 0, 1). \tag{85.33}$$

The derivatives $(\partial A_{ij}/\partial n_e)$ can be computed analytically. Using the fact that $\mathbf{A}^0\mathbf{n}^0 = \mathbf{B}^0$, we rewrite (85.31) as

$$\mathbf{A}^0\, \delta\mathbf{n} + [\mathbf{n}^0 \cdot (\partial\mathbf{A}/\partial n_e)^0 - (\partial\mathbf{B}/\partial n_e)^0]\, \delta n_e = 0. \tag{85.34}$$

Further, from (85.27) we have

$$\sum_{i=1}^{L} \delta n_i + \delta n_p + \delta n_e = N - \left(\sum_{i=1}^{L} n_i^0 + n_p^0 + n_e^0\right). \tag{85.35}$$

Equations (85.34) and (85.35) provide a linear system yielding $\delta\mathbf{n}$ and δn_e. Solving them, we revise all populations and recompute $\mathbf{A}$ and $\mathbf{B}$ using the new value of n_e, and iterate until $|\delta n_i/n_i|$ is less than some small number for all n_i and n_e. Convergence is quadratic, giving precise results in a few iterations if the original estimate of n_e^0 is reasonable. In some formulations one uses *departure coefficients*

$$b_i \equiv n_i / n_i^* \tag{85.36}$$

instead of occupation numbers per se.

As we will see in §6.7 and Chapters 7 and 8, the linearization procedure just described is a restrictive example of the general procedure used to solve the coupled equations of hydrodynamics, radiative transfer, and statistical equilibrium. In general we do not know T, N, and J_ν, but have only preliminary estimates; we then linearize (85.28) as

$$\delta \mathbf{n} = (\partial \mathbf{n}/\partial n_e)\, \delta n_e + (\partial \mathbf{n}/\partial T)\, \delta T + \sum_k (\partial \mathbf{n}/\partial J_k)\, \delta J_k \tag{85.37}$$

where the derivatives all have the form

$$(\partial \mathbf{n}/\partial x) = \mathbf{A}^{-1}[(\partial \mathbf{B}/\partial x) - \mathbf{n} \cdot (\partial \mathbf{A}/\partial x)]. \tag{85.38}$$

Analytical expressions can be written for $(\partial \mathbf{A}/\partial n_e)$, $(\partial \mathbf{A}/\partial T)$, and $(\partial \mathbf{A}/\partial J_k)$; see (**A7**), (**M2**), and (**M3**). In (85.38) all terms on the right-hand side are evaluated using current estimates of **A**, **B**, and **n**, the latter obtained from $\mathbf{An} = \mathbf{B}$. Equations (85.37) and (85.35) are then solved simultaneously with the linearized transfer and dynamical equations (see §88).

Finally, we outline the method for solving rate equations in a dynamical atmosphere. Assuming that all quantities are known at t^n, we wish to solve for new level populations at time $t^{n+1} = t^n + \Delta t^{n+(1/2)}$. In (85.24) we replace the zero right-hand side with the time derivative on the left-hand side of (85.5), represented by a finite difference, obtaining

$$[(\mathbf{n}^{n+1}/\rho^{n+1}) - (\mathbf{n}^n/\rho^n)]/\Delta t^{n-(1/2)} = \theta \mathbf{a}^{n+1}(\mathbf{n}^{n+1}/\rho^{n+1}) + (1-\theta)\mathbf{a}^n(\mathbf{n}^n/\rho^n). \tag{85.39}$$

where the superscripts indicate the time level at which quantities are evaluated. The parameter θ determines the centering of the time derivative. Intuitively one expects $\theta = \frac{1}{2}$ to be optimal because it weighs information at t^n and t^{n+1} equally in the integration over $\Delta t^{n+(1/2)}$. However, for most astrophysical problems the microscopic-process equilibrium times are orders of magnitude shorter than a dynamical timestep, hence level populations evolve through a sequence of quasistatic equilibria. Equations (85.39) are therefore very *stiff*, and for numerical stability one must choose $\theta = 1$, that is, a backward Euler or fully implicit scheme. Equations (85.39) can then be rewritten as

$$\tilde{\mathbf{a}}^{n+1}\mathbf{n}^{n+1} = \mathbf{b}^{n+1}, \tag{85.40}$$

where $b_i^{n+1} = (-\rho^{n+1}/\rho^n) n_i^n / \Delta t^{n+(1/2)}$, $(i = 1, \ldots, L)$, and $\tilde{\mathbf{a}}^{n+1}$ is the rate matrix with the term $-1/\Delta t^{n+(1/2)}$ added to the diagonal elements. Equations (85.26) and (85.27) remain unaltered, and are also evaluated at t^{n+1}. The system (85.40) plus (85.26) is again of the general form $\mathbf{An} = \mathbf{B}$, and the linearization scheme proceeds exactly as before, yielding equations of the general form (85.37). These are solved simultaneously with (85.35) and the linearized transfer and hydrodynamic equations (see §§88 and 104).

86. *Thermal Properties of a Nonequilibrium Gas*

In LTE, all occupation numbers and thermodynamic properties of the material are determined by two local state variables, say N and T (cf. §14). But for non-LTE, the rate equations imply that $n_i = n_i(N, T, J_\nu)$, where J_ν denotes the complete frequency spectrum of the mean intensity. Thus we cannot determine the particle distribution function until we know the photon distribution function; that is, we have as many new "state variables" as are required to specify the frequency distribution of the radiation field. Moreover, the radiation field is fundamentally nonlocal.

Thus when there are departures from LTE we cannot hope to express thermodynamic properties of the material as a function of local state variables. At best we can write some properties in terms of particle densities. For example, we can express the internal energy of the material (assumed to be pure hydrogen) as

$$\rho e = \sum_{i=1}^{L} n_i \varepsilon_i + n_p \varepsilon_I + \tfrac{3}{2} NkT \tag{86.1}$$

where ε_I is the ionization energy of hydrogen, and $\varepsilon_i = (1 - i^{-2})\varepsilon_I$. Similarly the specific enthalpy is

$$\rho h = \sum_{i=1}^{L} n_i \varepsilon_i + n_p \varepsilon_I + \tfrac{5}{2} NkT. \tag{86.2}$$

We used this approach in §75 to write formulae for the opacity and emissivity.

For other quantities, such as entropy, we can go back to basic definitions such as (11.1) and (11.2), but their usefulness is problematical. In yet other cases it is not clear how a quantity can be defined; for example, we cannot write formulae for c_v and c_p because the gas responds differently when the energy input is radiative than when it is thermal energy of microscopic motions. Some quantities are simply meaningless; for example, the adiabatic exponents no longer make sense because, in the presence of a nonequilibrium radiation field, the material is inherently nonadiabatic owing to radiative energy exchange from one point in the medium to another.

It is sometimes useful to have rough models for the nonequilibrium thermodynamic properties of material, for example, in analyses of linear wave propagation or local stability. For example, one might assume an *imposed* radiation field and fixed radiative rates, and then perturb the rate equations with respect to, say, δT and δp, and use the δn's to evaluate such quantities as $\delta \chi_\nu$ and $\delta \eta_\nu$ (useful in the linearized transfer equation), or δe (for studies of energy balance). A cruder model would be to assume all the b factors for bound levels to be fixed, and then calculate δn_i as $b_i\, \delta n_i^*$ for perturbations δT and δp. But we stress that these approaches are not rigorous, and the only completely satisfactory treatment is to enforce a self-consistent nonlocal coupling among the fluid equations, rate equations, and transfer equations from the outset.

6.7 Solution of the Coupled Transfer and Statistical Equilibrium Equations in Static Media

We are now in a position to combine the transfer and statistical equilibrium equations with the goal of obtaining consistency between the radiation field and material properties, subject to constraints of momentum and energy balance, which is central to several classes of astrophysical problems: (1) In the static non-LTE *model atmospheres problem* we attempt to determine simultaneously the radiation field (from the transfer equation), the state of the material (from the statistical equilibrium equations), and the structure of the medium (from constraints of hydrostatic and radiative equilibrium). We focus closely on this problem in this chapter because it provides a good framework for the development of the basic formalism. A simpler problem is to assume LTE, replacing the statistical equilibrium equations with the Saha-Boltzmann relation. (2) In the *statistical equilibrium problem* we solve the combined transfer and statistical equilibrium equations assuming that the atmospheric structure is known. Here one includes many levels and transitions so as to simulate an observed spectrum. We will not discuss this class of problem; see (**M2,** Chaps. 10–12). (3) In the *dynamical model atmospheres problem* we solve the transfer equation, time-dependent statistical equilibrium equations, and the hydrodynamical equations. Again the problem can be simplified by assuming LTE. Such problems are considered in Chapters 7 and 8. (4) Finally, one may do a statistical equilibrium problem in a time-varying atmosphere, for example, to calculate the spectrum of a variable star. Such problems are outside the scope of this book.

87. *The Two-Level Atom*

The two characteristic features of non-LTE transfer problems are that (1) the source function (for both lines and continua) contains a dominant scattering term and (2) the radiation field in one transition affects the radiation field in other transitions. To gain physical insight we consider a highly simplified atomic model, the *two-level atom*, consisting of two bound levels and a continuum, for which we can obtain an analytical expression for the source function that displays the scattering term explicitly, and shows the nature of the other source-sink terms. We can then see how atomic processes affect the thermalization of the radiation field in the line joining the two levels, and armed with this insight we can appreciate the implications of the computational strategy developed in §88. To supplement our brief discussion see (**M2,** Chap. 11).

THE LINE SOURCE FUNCTION

Consider the spectrum line formed between the lower and upper levels, l and u, of an atom. Assume that the line is so strong (e.g., a resonance line) that we need consider only the opacity and emissivity of the line itself. From (73.10), the average line opacity is

$$\chi_{lu} = (B_{lu} h\nu_{lu}/4\pi)[n_l - (g_l/g_u)n_u]. \tag{87.1}$$

Let $d\tau \equiv -\chi_{lu}\,dz$; the transfer equation is then

$$\mu(dI_\nu/d\tau) = \phi_\nu(I_\nu - S_l), \tag{87.2}$$

where the source function is [cf. (73.13)]

$$S_l = (2h\nu_{lu}^3/c^2)[(g_u n_l/g_l n_u) - 1]^{-1}. \tag{87.3}$$

The occupation numbers of levels l and u are determined by the two rate equations

$$n_l(B_{lu}\bar{J}_{lu} + C_{lu} + R_{l\kappa} + C_{l\kappa}) = n_u(A_{ul} + B_{ul}\bar{J}_{lu} + C_{ul}) + n_l^*(R_{\kappa l} + C_{l\kappa}) \tag{87.4}$$

and

$$n_u(A_{ul} + B_{ul}\bar{J}_{lu} + C_{ul} + R_{u\kappa} + C_{u\kappa}) = n_l(B_{lu}\bar{J}_{lu} + C_{lu}) + n_u^*(R_{\kappa u} + C_{u\kappa}), \tag{87.5}$$

where $\bar{J}_{lu} \equiv \int \phi_\nu J_\nu\, d\nu$. From (87.4) and (87.5) we derive an explicit analytical expression for S_l by solving for the ratio (n_l/n_u), substituting into (87.3), and using the Einstein relations (73.8) and (73.9); after some algebra one obtains

$$S_l = [\bar{J}_{lu} + \varepsilon B_\nu(T_e) + \theta]/(1 + \varepsilon + \eta), \tag{87.6}$$

where

$$\varepsilon \equiv C_{ul}(1 - e^{-h\nu_{lu}/kT})/A_{ul}, \tag{87.7}$$

$$\eta \equiv \frac{(R_{u\kappa} + C_{u\kappa})n_l^*(R_{\kappa l} + C_{l\kappa}) - g_l(R_{l\kappa} + C_{l\kappa})n_u^*(R_{\kappa u} + C_{u\kappa})/g_u}{A_{ul}[n_l^*(R_{\kappa l} + C_{l\kappa}) + n_u^*(R_{\kappa u} + C_{u\kappa})]}, \tag{87.8}$$

and

$$\theta \equiv \left(\frac{2h\nu_{lu}^3}{c^2}\right)\left(\frac{g_l}{g_u A_{ul}}\right)\frac{(R_{l\kappa} + C_{l\kappa})n_u^*(R_{\kappa u} + C_{u\kappa})}{[n_l^*(R_{\kappa l} + C_{l\kappa}) + n_u^*(R_{\kappa u} + C_{u\kappa})]}. \tag{87.9}$$

Equation (87.6), first derived by R. N. Thomas (**T1**), has played an important role in line-formation theory.

Each term in (87.6) has a simple physical interpretation. Consider first the denominator, which contains *sink terms*, all normalized to the spontaneous emission rate. The first term corresponds to scattered photons. The second is the rate of collisional de-excitation of the upper level, hence accounts for collisional destruction and thermalization of line photons. The third is proportional to the total ionization rate from the upper level times the fraction of recombinations into the lower level, hence accounts for destruction of line photons by processes that transfer electrons from the upper to the lower state via the continuum.

The numerator contains *source terms*. The first is a noncoherent scattering reservoir of line photons resulting from the cumulative contributions of sources and sinks over an interaction volume. The second is the thermal source, representing photons created by collisional excitation of an electron to the upper level, followed by radiative decay; this term depends explicitly on the local electron kinetic temperature T_e. The third term is proportional to the total ionization rate from the ground level times the fraction of recombinations into the upper level, hence accounts for line photons created by continuum processes that transfer electrons from the lower to upper level, whence they radiatively decay. The quantity θ can be rewritten as $\theta = \eta B^*$, where

$$B^* = \left(\frac{2h\nu_{lu}^3}{c^2}\right)\left\{\left(\frac{n_l^* g_u}{n_u^* g_l}\right)\left[\frac{(R_{u\kappa} + C_{u\kappa})(R_{\kappa l} + C_{l\kappa})}{(R_{l\kappa} + C_{l\kappa})(R_{\kappa u} + C_{u\kappa})}\right] - 1\right\}^{-1}. \qquad (87.10)$$

We can then express B^* as $B_\nu(T_R)$ where T_R is a characteristic radiation temperature that can exceed, or be less than, T_e depending on the relative sizes of photoionization and recombination rates.

At great depth in an atmosphere, large densities imply high collision rates, ε becomes large, and S_l thermalizes. If the continua are also opaque so that $J_\nu \to B_\nu$, then $R_{l\kappa} \to R_{l\kappa}^* = R_{\kappa l}^*$, while $R_{u\kappa} \to R_{u\kappa}^* = R_{\kappa u}^*$, and $B^* \to B_\nu(T_e)$. Hence at depth $S_l \to B_\nu(T_e)$. The behavior of S_l near the boundary depends on the relative sizes of the source-sink terms; in general both ε and η will be much smaller than unity. If $\varepsilon > \eta$ and $\varepsilon B > \eta B^*$, the line is *collision dominated*; S_l then tends to couple to the local electron temperature T_e. If $\eta > \varepsilon$ and $\eta B^* > \varepsilon B$, the line is *photoionization dominated*; here S_l tends to couple to the radiation temperature T_R. *Mixed domination* lines have $\varepsilon > \eta$ but $\eta B^* > \varepsilon B$ or vice versa. These three cases can behave quite differently in a stellar chromosphere, see (**M2,** Chap. 11).

THE LINE TRANSFER EQUATION

Given a model atmosphere, one can compute the depth variation of ε, η, $B_\nu(T_e)$, and B^*. Substituting (87.6) into (87.2) we have a noncoherent scattering problem with known coefficients, which can readily be solved by the numerical techniques described in §83. While such a solution certainly yields the answer, it reveals little about the underlying physics. To gain further insight let us therefore consider a simpler problem, and suppose that $\eta = \theta \equiv 0$, and that both ε and B are constant. The transfer equation is then

$$\mu(\partial I_\nu/\partial \tau) = \phi_\nu[I_\nu - (1-\tilde{\varepsilon})\bar{J} - \tilde{\varepsilon}B], \qquad (87.11)$$

where $\tilde{\varepsilon} \equiv \varepsilon/(1+\varepsilon)$. Equation (87.11) bears a strong resemblance to the archetype scattering problem (83.1) to (83.6) except that now the scattering is noncoherent. We thus expect that the radiation field in a line will behave as described in §83, that is, that $\bar{J}$ will depart markedly from B at the surface, and that this departure will extend over a thermalization depth $\mathscr{L} \gg 1$ in line optical-depth units.

Detailed analysis [see e.g., (**M2,** §11–2)] shows that these expectations are met. In particular one finds that

$$S_l(0) = \varepsilon^{1/2}B, \tag{87.12}$$

in close analogy with (83.8). The thermalization depth differs from our earlier result because the noncoherence of the scattering process implies that photons that would have been trapped in the line core at depths where the core is opaque can instead get redistributed into the transparent line wings, whence they escape. We can estimate the effects of this escape mechanism on S_l by the following argument.

THE THERMALIZATION DEPTH

Suppose the line has a Doppler profile. Measure frequencies within the line in Doppler units, that is, $x \equiv (\nu - \nu_0)/\Delta\nu_D$ where ν_0 is the line-center frequency, and the *Doppler width* is $\Delta\nu_D = (\nu_0/c)(2kT/m)^{1/2}$. Then the profile function is

$$\phi(x) = \pi^{-1/2}e^{-x^2}. \tag{87.13}$$

The escape probability for a photon of frequency x traveling along a ray with angle-cosine μ from a point at line optical depth τ is $\exp(-\tau\phi_x/\mu)$. Weighting escapes by ϕ_x, the probability of emission at x, we find the total *photon escape probability*

$$\begin{aligned} P_e(\tau) &= \tfrac{1}{2}\int_{-\infty}^{\infty} dx\,\phi_x \int_0^1 d\mu e^{-\tau\phi_x/\mu} = \tfrac{1}{2}\int_{-\infty}^{\infty} dx\,\phi_x \int_1^{\infty} dy\, y^{-2}e^{-\tau\phi_x y} \\ &= \tfrac{1}{2}\int_{-\infty}^{\infty} E_2(\tau\phi_x)\phi_x\, dx. \end{aligned} \tag{87.14}$$

Define x_1 such that $\tau\phi(x_1) = 1$. Then for $\tau \gg 1$, $E_2(\tau\phi_x)$ is approximately zero when $|x| \lesssim x_1$ because then $\tau\phi_x \gg 1$, and is approximately unity when $|x| \gtrsim x_1$ because $\tau\phi_x \ll 1$. We can thus approximate P_e as

$$P_e(\tau) \approx \int_{x_1}^{\infty} \phi_x\, dx, \tag{87.15}$$

which shows that P_e is determined by photon escapes in the line wing.

When $\tau \gg 1$, $x_1 \gg 1$. In this limit (87.15) yields

$$P_e(\tau) = \tfrac{1}{2}\operatorname{erfc}(x_1) \approx e^{-x_1^2}/(2\pi^{1/2}x_1) \tag{87.16}$$

for a Doppler profile, for which, from (87.13)

$$x_1 = [\ln(\tau/\pi^{1/2})]^{1/2}, \tag{87.17}$$

so that

$$P_e(\tau) \approx k/[\tau(\ln\tau)^{1/2}], \tag{87.18}$$

where k is a constant of order unity.

The escape probability is to be compared with P_d, the *photon destruction probability* per scattering. Ignoring continuum terms, $P_d = (C_{ul}/A_{ul}) = \varepsilon$ for a two-level atom. If $P_e \ll P_d$, the photon will surely be thermalized before it escapes and therefore $S_l \rightarrow B$; if $P_e \gg P_d$ photons escape freely, hence S_l can depart from B, perhaps drastically. It is thus reasonable to estimate the *thermalization length* $\mathscr{L}$ as the depth at which $P_e(\mathscr{L}) \approx P_d$. For a Doppler profile this gives

$$\mathscr{L} \sim k/\varepsilon \tag{87.19}$$

where k is a factor of order unity. We again find that the range over which S_l can differ markedly from B can be enormous when ε is small, as it usually is near a boundary surface. For coherent scattering we found earlier that $\mathscr{L} \sim \varepsilon^{-1/2}$; the larger value predicted by (87.19) results from the effects of noncoherence. For other line profiles $\mathscr{L}$ can be even larger; for example, for a Lorentz profile $\mathscr{L} \sim \varepsilon^{-2}$.

OVERLAPPING CONTINUUM

We assumed above that all opacity and emissivity comes from the line itself. When there is an overlapping continuum, which we assume is in LTE, the source function is given by (77.11), and the transfer equation becomes

$$\mu(\partial I_\nu/\partial\tau) = \phi_\nu[I_\nu - (1-\xi_\nu)\bar{J} - \xi_\nu B_\nu]. \tag{87.20}$$

Detailed analysis shows [see e.g., (**M2,** 350–354)] that the behavior of S_l is then determined by the average thermal coupling parameter

$$\bar{\xi} \equiv \int_{-\infty}^{\infty} \phi_x \xi_x \, dx = \delta + \varepsilon(1-\delta), \tag{87.21}$$

where

$$\delta \equiv r \int_{-\infty}^{\infty} \phi_x(\phi_x + r)^{-1} \, dx. \tag{87.22}$$

Physically, δ is the continuum destruction probability: it is the profile-weighted average of the probability $r/(\phi + r)$ that a line photon will be absorbed by the continuum, times the (unit) probability that the absorbed photon is thermalized. The continuum processes set a floor on $\bar{\xi}$, often much larger than ε itself, and guarantee that S_l will be thermalized when the continuum optical depth is unity or larger, even if the line thermalization processes represented by ε would not by themselves force $S_l \rightarrow B$.

TRANSFER IN A RESONANCE CONTINUUM

The preceding analysis can be adapted to continua, say the Lyman continuum of hydrogen. We adopt the simplest possible atomic model: one bound state plus a continuum. We assume that the continuum is so opaque that we can ignore all other sources of opacity and emissivity. Then from

(75.1) and (75.2)

$$\chi_\nu = (n_1 - n_1^* e^{-h\nu/kT})\alpha_\nu = (b_1 - e^{-h\nu/kT}) n_1^* \alpha_\nu, \tag{87.23}$$

while

$$\eta_\nu = n_1^* \alpha_\nu B_\nu (1 - e^{-h\nu/kT}). \tag{87.24}$$

For the Lyman continuum $h\nu/kT \gg 1$ (otherwise the hydrogen is mostly ionized and the assumptions just stated fail), hence we can neglect $e^{-h\nu/kT}$. We then have

$$S_\nu = \eta_\nu/\chi_\nu = B_\nu/b_1, \tag{87.25}$$

and the transfer equation to be solved is

$$\mu(\partial I_\nu/\partial\tau) = \phi_\nu[I_\nu - (B_\nu/b_1)]; \tag{87.26}$$

here we have written $\alpha_\nu \equiv \alpha_0 \phi_\nu$ and $d\tau \equiv -\chi_0\,dz = -n_1\alpha_0\,dz$ where α_0 is the cross section at the continuum head.

From the rate equation for a one-level atom, we find that b_1 is given by

$$b_1 = \left[4\pi \int_{\nu_0}^{\infty} (\alpha_\nu/h\nu) B_\nu\,d\nu + C_{1\kappa}\right] \bigg/ \left[4\pi \int_{\nu_0}^{\infty} (\alpha_\nu/h\nu) J_\nu\,d\nu + C_{1\kappa}\right], \tag{87.27}$$

where we ignored stimulated emission. Because $(h\nu/kT) \gg 1$, the factor $\exp(-h\nu/kT)$ in B_ν decays away rapidly from its value at $\nu = \nu_0$. Thus the two integrands in (87.27) peak sharply at $\nu \approx \nu_0$, and to a good approximation we can replace the integrals by $4\pi w_0(\alpha_0/h\nu_0)J_0$ and $4\pi w_0(\alpha_0/h\nu_0)B_0$, where w_0 is an appropriate quadrature weight. We then consider the transfer problem only at the continuum head. If we write $\varepsilon = C_{1\kappa}/[(4\pi w_0 \alpha_0 B_0/h\nu_0) + C_{1\kappa}]$, the source function at ν_0 is

$$S_0 = (1-\varepsilon)J_0 + \varepsilon B_0. \tag{87.28}$$

and the transfer equation at ν_0, in the Eddington approximation, becomes

$$\tfrac{1}{3}(d^2 J_0/d\tau_0^2) = \varepsilon(J_0 - B_0). \tag{87.29}$$

Equation (87.29) is identical to the archetype problem (83.6), hence we know that the thermalization depth will be $\sim\varepsilon^{-1/2}$, while at the boundary $S_0 \sim \varepsilon^{1/2} B_0$, which implies $b_1 \sim \varepsilon^{-1/2}$. In a stellar atmosphere $\varepsilon \ll 1$, hence in the Lyman continuum there are large departures from LTE, which persist to great optical depth. Thus both lines and continua are subject to major non-LTE effects.

THE MULTILEVEL PROBLEM

The atomic models discussed above are drastically oversimplified; to achieve realism we must consider multilevel atoms. In the multilevel case we might try to follow the approach used above, manipulating the rate equation to obtain an analytical expression of the form

$$S_{ij} = (\bar{J}_{ij} + \alpha_{ij})/(1 + \beta_{ij}) \tag{87.30}$$

for the source function in each transition $i \to j$. This approach is called the *equivalent two-level atom* (ETLA) *method* because (87.30) is of the same general form as (87.6). There are many different formulations of this method, which use different groupings of terms within α_{ij} and β_{ij}, and different methods of solving the equations [see, e.g., (**A3,** Chap. 3), (**J1,** Chaps. 6 and 8), (**M2,** §12–1), and (**T2,** Chap. 3)]. Given source functions of the form (87.30), it is straightforward to solve the transfer equation in each transition.

Superficially this procedure looks much simpler than it is. The fundamental problem is that the coefficients α_{ij} and β_{ij} contain terms that depend on the radiation fields in all transitions other than $(i \to j)$ [e.g., for a two-level atom (87.6) explicitly displays the dependence of S_l on the radiative rates in the continua $(l \to \kappa)$ and $(u \to \kappa)$]. Thus we cannot in fact specify the source-sink terms for any S_{ij} until we know the radiation field in every other transition. But we can't compute these radiation fields until we know the relevant source functions. We must therefore use some kind of iteration scheme [see e.g., (**A3,** Chaps. 4 and 7), (**T2,** Chaps. 5 and 6)]. In some cases the iteration will converge; in others it fails, forcing one to seek clever schemes (often based on trial and error) to accelerate convergence.

In the face of the difficulties just described, Jefferies (**J1,** Chap. 8) stressed the conceptual importance of viewing radiation in the entire transition array of an atom as belonging to a *collective photon pool*, thus recognizing that photons do not belong uniquely to any one transition, but interlock all transitions in a manner prescribed by the equations of statistical equilibrium. From this standpoint one sees that it is essential to treat all transitions *simultaneously*; let us now address this problem in the context of constructing a static non-LTE model atmosphere.

88. *The Complete Linearization Method*

In the non-LTE model atmospheres problem we wish to determine, at each point in the medium, the frequency distribution of the radiation field, the distribution of atoms and ions over their bound states, and the temperature and density. The imposed constraints are momentum balance, energy balance, steady-state statistical equilibrium, and charge and number conservation. Our discussion will be brief; further details are given in (**M2,** Chaps. 7 and 12).

BASIC EQUATIONS

The pressure distribution is determined by the equation of hydrostatic equilibrium $\nabla p = \mathbf{f}$, where $\mathbf{f}$ is the total force per unit volume acting on the material; cf. (23.25). For a one-dimensional planar medium

$$(dp/dm) = g - (f_R/\rho), \tag{88.1}$$

where $dm \equiv -\rho\, dz$ and, from (78.12),

$$(f_R/\rho) = (4\pi/c) \int_0^\infty \omega_\nu H_\nu \, d\nu, \tag{88.2}$$

where $\omega_\nu \equiv \chi_\nu/\rho$. Using (78.17) we can rewrite (88.1) and (88.2) as

$$(dp/dm) + (dP/dm) = g, \tag{88.3}$$

where

$$P = \int_0^\infty P_\nu \, d\nu = (4\pi/c) \int_0^\infty f_\nu J_\nu \, d\nu. \tag{88.4}$$

The temperature distribution is determined by the requirement of radiative equilibrium:

$$4\pi \int_0^\infty (\eta_\nu - \chi_\nu J_\nu) \, d\nu = 0. \tag{88.5}$$

Assuming thermal emission and coherent electron scattering, $\chi_\nu = \kappa_\nu + n_e\sigma_e$ and $\eta_\nu = \eta_\nu^t + n_e\sigma_e J_\nu$, hence (88.5) becomes

$$4\pi \int_0^\infty (\eta_\nu^t - \kappa_\nu J_\nu) \, d\nu = 0, \tag{88.6}$$

where η_ν^t and κ_ν are given by (75.1) and (75.2).

The state of the material is determined by the equations of statistical equilibrium, charge conservation, and number conservation, as in (85.25) to (85.27).

The only radiation quantity appearing in these equations is J_ν, hence we use the transfer equation in angle-integrated form:

$$\frac{\partial^2(f_\nu J_\nu)}{\partial \tau_\nu^2} = J_\nu - S_\nu = \left(1 - \frac{n_e\sigma_e}{\chi_\nu}\right) J_\nu - \frac{\eta_\nu^t}{\chi_\nu}. \tag{88.7}$$

DISCRETIZATION

Equations (88.3), (88.6), (88.7), and (85.25) to (85.27) completely specify the problem. We discretize them for a set of frequencies $\{\nu_k\}$, and mass shells with boundaries at $\{m_d\}$ as in §83. The discretized form of (88.3) is

$$\begin{aligned} k[N_{d+(1/2)}T_{d+(1/2)} - N_{d-(1/2)}T_{d-(1/2)}] \\ + (4\pi/c) \sum_{k=1}^{K} w_k [f_{d+(1/2),k} J_{d+(1/2),k} - f_{d-(1/2),k} J_{d-(1/2),k}] \\ = g\, \Delta m_d, \qquad (d = 2, \ldots, D), \end{aligned} \tag{88.8}$$

where $\Delta m_d \equiv \frac{1}{2}[\Delta m_{d-(1/2)} + \Delta m_{d+(1/2)}]$, and w_k is a frequency-quadrature weight. To obtain a starting value at $d = 1$ we use (88.1), assuming that the material is transparent (so that H_ν, hence f_R, is fixed from the boundary outward). We have

$$N_{3/2} k T_{3/2} = m_1 \left[g - (4\pi/c) \sum_{k=1}^{K} w_k \omega_{3/2k} \mathscr{h}_k \mathscr{j}_k J_{3/2k} \right], \tag{88.9}$$

where the geometric factors $\mathscr{h}_k$ and $\mathscr{j}_k$ are defined by (83.44) and (83.45).

The discrete representation of (88.6) is

$$\sum_{k=1}^{K} w_k[\eta^t_{d+(1/2),k} - \kappa_{d+(1/2),k} J_{d+(1/2),k}] = 0, \qquad (d = 1, \ldots, D), \tag{88.10}$$

and the transfer equation and its boundary conditions are given by (83.58) to (83.61). The material equations can be used exactly as written in §85.

METHOD OF SOLUTION

In solving the system posed above we must bear in mind several important points. First, no variable is more fundamental than any other because all interact; thus within the slab $\Delta m_{d+(1/2)}$ the solution vector to be found is

$$\boldsymbol{\psi}_{d+(1/2)} = [J_{d+(1/2),1}, \ldots, J_{d+(1/2),K}, N, T, n_e, n_1, \ldots, n_L], \tag{88.11}$$

where L is the total number of levels including ions. Some of the information in $\boldsymbol{\psi}_{d+(1/2)}$ is, strictly speaking, redundant; in particular the J's determine the n's via the rate equations. But because we wish to know all these quantities we retain them in the formulation.

Second, all variables are strongly coupled globally via radiative transfer. A change in *any* variable at *any* point in the medium implies changes in *all* variables at *all* other points. The method of solution must account for this global coupling.

Third, the system is nonlinear, and must be solved by iteration.

Assume that the desired solution $\boldsymbol{\psi}_{d+(1/2)}$ can be written $\boldsymbol{\psi}_{d+(1/2)} = \boldsymbol{\psi}^0_{d+(1/2)} + \delta\boldsymbol{\psi}_{d+(1/2)}$ where $\boldsymbol{\psi}^0_{d+(1/2)}$ is the current (approximate) solution. The system to be solved at each depth point is of the form $\mathbf{f}_{d+(1/2)}(\boldsymbol{\psi}) = 0$, so we demand that $\delta\boldsymbol{\psi}$ be such that $\mathbf{f}_{d+(1/2)}(\boldsymbol{\psi}^0 + \delta\boldsymbol{\psi}) = 0$. We linearize this system and demand that

$$\mathbf{f}_{d+(1/2)}(\boldsymbol{\psi}^0) + \sum_j [\partial \mathbf{f}_{d+(1/2)}/\partial \psi_j]\, \delta\psi_j = 0, \qquad (d = 1, \ldots, D), \tag{88.12}$$

where j runs over all variables. These equations determine the $\delta\boldsymbol{\psi}$'s; let us examine them in more detail.

The linearized equation of hydrostatic equilibrium is

$$\begin{aligned} k[T_{d+(1/2)}\, \delta N_{d+(1/2)} &+ N_{d+(1/2)}\, \delta T_{d+(1/2)} \\ &- T_{d-(1/2)}\, \delta N_{d-(1/2)} - N_{d-(1/2)}\, \delta T_{d-(1/2)}] \\ &+ (4\pi/c) \sum_{k=1}^{K} w_k[f_{d+(1/2),k}\, \delta J_{d+(1/2),k} - f_{d-(1/2),k}\, \delta J_{d-(1/2),k}] \end{aligned} \tag{88.13}$$

$$\begin{aligned} = g\, \Delta m_d &- N_{d+(1/2)} k T_{d+(1/2)} + N_{d-(1/2)} k T_{d-(1/2)} \\ &- (4\pi/c) \sum_{k=1}^{K} w_k[f_{d+(1/2),k} J_{d+(1/2),k} - f_{d-(1/2),k} J_{d-(1/2),k}], \end{aligned}$$

with a similar equation for the linearized boundary condition (88.9). The linearized radiative equilibrium equation is

$$\begin{aligned}\sum_{k=1}^{K} w_k[\kappa_{d+(1/2),k}\,\delta J_{d+(1/2),k}+J_{d+(1/2),k}\,\delta\kappa_{d+(1/2),k}-\delta\eta^t_{d+(1/2),k}]&\\ =\sum_{k=1}^{K} w_k[\eta^t_{d+(1/2),k}-\kappa_{d+(1/2),k}J_{d+(1/2),k}]&\end{aligned}\tag{88.14}$$

where from (75.1) we express $\delta\kappa$ as

$$\delta\kappa=(\partial\kappa/\partial T)\,\delta T+(\partial\kappa/\partial n_e)\,\delta n_e+\sum_{l=1}^{L}(\partial\kappa/\partial n_l)\,\delta n_l,\tag{88.15}$$

and similarly for $\delta\eta^t$.

The linearized transfer equations are

$$\begin{aligned}&\frac{f_{d-(1/2),k}\,\delta J_{d-(1/2),k}}{\Delta\tau_{dk}\,\Delta\tau_{d+(1/2),k}}+\frac{f_{d+(3/2),k}\,\delta J_{d+(3/2),k}}{\Delta\tau_{d+(1/2),k}\,\Delta\tau_{d+1,k}}\\ &\quad-\left\{\frac{f_{d+(1/2),k}}{\Delta\tau_{d+(1/2),k}}\left(\frac{1}{\Delta\tau_{dk}}+\frac{1}{\Delta\tau_{d+1,k}}\right)+\left[1-\frac{n_{e,d+(1/2)}\sigma_e}{\chi_{d+(1/2),k}}\right]\right\}\delta J_{d+(1/2),k}\\ &\quad+a_{d+(1/2),k}\,\delta\omega_{d-(1/2),k}+b_{d+(1/2),k}\,\delta\omega_{d+(1/2),k}+c_{d+(1/2),k}\,\delta\omega_{d+(3/2),k}\\ &\quad-[\eta^t_{d+(1/2),k}+n_{e,d+(1/2)}\sigma_e J_{d+(1/2),k}]\frac{\delta\chi_{d+(1/2),k}}{\chi^2_{d+(1/2),k}}\\ &\quad+\frac{\delta\eta^t_{d+(1/2),k}}{\chi_{d+(1/2),k}}+\frac{\sigma_e J_{d+(1/2),k}}{\chi_{d+(1/2),k}}\,\delta n_{e,d+(1/2)}\\ &=\beta_{d+(1/2),k}+J_{d+(1/2),k}-\frac{[n_{e,d+(1/2)}\sigma_e J_{d+(1/2),k}+\eta^t_{d+(1/2),k}]}{\chi_{d+(1/2),k}}\end{aligned}\tag{88.16}$$

where

$$\alpha_{d+(1/2),k}\equiv[f_{d+(1/2),k}J_{d+(1/2),k}-f_{d-(1/2),k}J_{d-(1/2),k}]/[\Delta\tau_{d+(1/2),k}\,\Delta\tau_{dk}],\tag{88.17}$$

$$\gamma_{d+(1/2),k}\equiv[f_{d+(1/2),k}J_{d+(1/2),k}-f_{d+(3/2),k}J_{d+(3/2),k}]/[\Delta\tau_{d+(1/2),k}\,\Delta\tau_{d+1,k}],\tag{88.18}$$

$$\beta_{d+(1/2),k}\equiv\alpha_{d+(1/2),k}+\gamma_{d+(1/2),k},\tag{88.19}$$

$$\begin{aligned}a_{d+(1/2),k}\equiv\{\alpha_{d+(1/2),k}+\tfrac{1}{2}\beta_{d+(1/2),k}[\Delta\tau_{dk}/\Delta\tau_{d+(1/2),k}]\}&\\ /[\omega_{d-(1/2),k}+\omega_{d+(1/2),k}],&\end{aligned}\tag{88.20}$$

$$\begin{aligned}c_{d+(1/2),k}\equiv\{\gamma_{d+(1/2),k}+\tfrac{1}{2}\beta_{d+(1/2),k}[\Delta\tau_{d+1,k}/\Delta\tau_{d+(1/2),k}]\}&\\ /[\omega_{d+(1/2),k}+\omega_{d+(3/2),k}],&\end{aligned}\tag{88.21}$$

and

$$b_{d+(1/2),k}\equiv a_{d+(1/2),k}+c_{d+(1/2),k}.\tag{88.22}$$

The density is

$$\rho = (N - n_e) m_H \sum_i \alpha_i A_i \equiv (N - n_e)\bar{m} \tag{88.23}$$

where α_i is the fractional abundance, by number, of element i, having atomic weight A_i; hence $\delta\rho$, which appears in $\delta\omega$, can be written $\delta\rho = \bar{m}(\delta N - \delta n_e)$.

Equations (88.13) to (88.23) plus the linearized statistical equilibrium, number conservation, and charge conservation equations (85.37) and (85.35) can be assembled into a system of the form

$$-\mathbf{A}_{d+(1/2)}\, \delta\boldsymbol{\psi}_{d-(1/2)} + \mathbf{B}_{d+(1/2)}\, \delta\boldsymbol{\psi}_{d+(1/2)} - \mathbf{C}_{d+(1/2)}\, \delta\boldsymbol{\psi}_{d+(3/2)} = \mathbf{L}_{d+(1/2)}, \tag{88.24}$$

which is solved by the Feautrier method (cf. §83). Here $\mathbf{L}_{d+(1/2)}$ is the residual error in the transfer and constraint equations for the current solution $\boldsymbol{\psi}^0_{d+(1/2)}$; as $\mathbf{L} \rightarrow 0$, the corrections $\delta\boldsymbol{\psi} \rightarrow 0$.

Mathematically the complete linearization method is merely a multidimensional Newton-Raphson iteration; at each stage the equations are internally consistent to first order. But it is extremely important to appreciate the system's physical content. Thus the linearized rate equations provide an algorithm that describes how photons are shuffled about within the collective photon pool in response to a change in material properties, or, reciprocally, how the material responds to a change in the radiation field. The linearized hydrostatic and radiative equilibrium equations describe how the pressure and temperature respond to local changes in the radiation field or material properties. Local changes are coupled to all other points in the medium via the tridiagonal linearized hydrostatic and transfer equations. This linearization correctly handles both local and global coupling, and experience shows it to be an effective tool for solving the problem.

The iteration scheme starts from an initial model constructed assuming LTE and adopting the grey temperature distribution on the Rosseland mean scale [cf. (82.23)]. The hydrostatic equation is integrated step by step for this $T(\tau_R)$, and from N and T one determines n_i^* for all atomic levels at all depths. Using the linearization procedure, the model is adjusted to give strict radiative equilibrium for the actual nongrey opacity, still assuming LTE. To obtain a non-LTE model, the Saha-Boltzmann relations are replaced by the statistical equilibrium equations.

After each linearization step, all occupation numbers are updated using the new temperature, density, and radiation field; new Eddington factors are then determined from a formal solution, and the procedure is iterated to convergence. If the Eddington factors were known and fixed, convergence would be quadratic; in practice the need to update Eddington factors slows the convergence rate, but errors usually diminish substantially at each iteration. Further details are given in (**M2,** Chap. 7), (**M3**), and the references cited therein. Note that for LTE models the Rybicki scheme is more efficient than the Feautrier scheme [see (**G1**) or (**M2,** 180–185)].

We now have all of the ingredients needed to describe radiative transfer and the interaction between radiation and material. In the next chapter we apply this knowledge to the dynamics of radiating fluids.

References

(A1) Abramowitz, M. and Stegun, I. (1964) *Handbook of Mathematical Functions.* Washington, D.C.: U.S. Department of Commerce.

(A2) Aller, L. H. (1963) *Astrophysics: The Atmospheres of the Sun and Stars.* (2nd ed.) New York: Ronald.

(A3) Athay, R. G. (1972) *Radiation Transport in Spectral Lines.* Dordrecht: Reidel.

(A4) Auer, L. H. (1967) *Astrophys. J. Letters,* **150,** L53.

(A5) Auer, L. H. (1971) *J. Quant. Spectrosc. Rad. Transf.,* **11,** 573.

(A6) Auer, L. H. (1976) *J. Quant. Spectrosc. Rad. Transf.,* **16,** 931.

(A7) Auer, L. H. and Mihalas, D. (1969) *Astrophys. J.,* **158,** 641.

(B1) Bhatnagar, P., Krook, M., Menzel, D. H., and Thomas, R. N. (1955) *Vistas in Astron.,* **1,** 296.

(B2) Böhm, K.-H. (1960) in *Stellar Atmospheres,* ed. J. L. Greenstein, Chapter 3. Chicago: University of Chicago.

(C1) Campbell, P. M. (1965) *Lawrence Livermore Laboratory Report No. UCRL-12411.* Livermore: University of California.

(C2) Campbell, P. M. (1969) *Int. J. Heat Mass Transf.,* **12,** 497.

(C3) Carbon, D. (1974) *Astrophys. J.,* **187,** 135.

(C4) Carlson, B. G. (1963) in *Methods in Computational Physics,* ed. B. Alder, S. Fernbach, and M. Rotenberg, Vol. **1,** Chapter 1. New York: Academic.

(C5) Chandrasekhar, S. (1939) *An Introduction to the Study of Stellar Structure.* Chicago: University of Chicago.

(C6) Chandrasekhar, S. (1950) *Radiative Transfer.* Oxford: Oxford University Press.

(C7) Cox, J. P. and Giuli, R. T. (1968) *Principles of Stellar Structure.* New York: Gordon and Breach.

(D1) Dragon, J. N. and Mutschlecner, J. P. (1980) *Astrophys. J.,* **239,** 1045.

(F1) Feautrier, P. (1964) *C. R. Acad. Sci. Paris,* **258,** 3189.

(F2) Freeman, B. E., Hauser, L. E., Palmer, J. T., Pickard, S. O., Simmons, G. M., Williston, D. G., and Zerkle, J. E. (1968). *DASA Report No. 2135,* Volume I. La Jolla: Systems, Science, and Software, Inc.

(G1) Gustafsson, B. and Nissen, P. (1972) *Astron. and Astrophys.,* **19,** 261.

(H1) House, L. L. and Steinitz, R. (1975) *Astrophys. J.,* **195,** 235.

(H2) Huang, K. (1963) *Statistical Mechanics.* New York: Wiley.

(H3) Hundhausen, A. J. (1972) *Coronal Expansion and Solar Wind.* New York: Springer.

(J1) Jefferies, J. T. (1968) *Spectral Line Formation.* Waltham: Blaisdell.

(K1) Kourganoff, V. (1952) *Basic Methods in Transfer Problems.* Oxford: Oxford University Press.

(K2) Krishna-Swamy, K. S. (1961) *Astrophys. J.,* **134,** 1017.

(K3) Kurucz, R. L., Peytremann, E., and Avrett, E. H. (1974) *Blanketed Models for Early-Type Stars*. Washington, D.C.: Smithsonian Institution.

(L1) Landi Degl'innocenti, E. (1983) *Solar Physics*, **85,** 3.

(L2) Landi Degl'innocenti, E. and Landi Degl'innocenti, M. (1972) *Solar Phys.*, **27,** 319.

(L3) Landi Degl'innocenti, E. and Landi Degl'innocenti, M. (1975) *Nuovo Cimento*, **27B,** 134.

(L4) Lathrop, K. D. and Carlson, B. G. (1967) *J. Comp. Phys.*, **2,** 173.

(L5) Lathrop, K. D. and Carlson, B. G. (1971) *J. Quant. Spectrosc. Rad. Transf.*, **11,** 921.

(L6) Lund, C. M. and Wilson, J. R. (1980) *Lawrence Livermore Laboratory Report No. UCRL-84678*. Livermore: University of California.

(M1) Mihalas, D. (1965) *Astrophys. J.*, **141,** 564.

(M2) Mihalas, D. (1978) *Stellar Atmospheres*. (2nd ed.) San Francisco: Freeman.

(M3) Mihalas, D., Auer, L. H., and Heasley, J. N. (1975) *NCAR Technical Note No. TN/STR-104*. Boulder: National Center for Atmospheric Research.

(M4) Milne, E. A. (1924) *Phil. Mag.*, **47,** 209.

(M5) Milne, E. A. (1966) in *Selected Papers on Physical Processes in Ionized Plasmas*, ed. D. H. Menzel, p. 77. New York: Dover.

(M6) Minerbo, G. (1978) *J. Quant. Spectrosc. Rad. Transf.*, **20,** 541.

(N1) Nelson, R. (1967) *Lawrence Livermore Laboratory Report No. TPNU 66–8*. Livermore: University of California.

(O1) Osborne, R. K. and Klevans, E. H. (1961) *Ann. Phys.*, **15,** 105.

(P1) Pai, S.-I. (1966) *Radiation Gas Dynamics*. New York: Springer.

(P2) Peraiah, A. and Grant, I. P. (1973) *J. Inst. Maths. Applics.*, **12,** 75.

(P3) Pomraning, G. C. (1973) *The Equations of Radiation Hydrodynamics*. Oxford: Pergamon.

(R1) Rybicki, G. B. (1971) *J. Quant. Spectrosc. Rad. Transf.*, **11,** 589.

(S1) Sampson, D. H. (1965) *Radiative Contributions to Energy and Momentum Transport in a Gas*. New York: Interscience.

(S2) Slater, J. C. (1960) *Quantum Theory of Atomic Structure*. New York: McGraw-Hill.

(S3) Sneden, C., Johnson, H. R., and Krupp, B. M. (1976) *Astrophys. J.*, **204,** 281.

(S4) Spillman, G. R. (1968) *Formulation of the Eddington Factor for Use in One-Dimensional Nonequilibrium Diffusion Calculations*. Los Alamos Scientific Laboratory Office Memorandum, January 25, 1968.

(S5) Sykes, J. B. (1951) *Mon. Notices Roy. Astron. Soc.*, **111,** 377.

(T1) Thomas, R. N. (1957) *Astrophys. J.*, **125,** 260.

(T2) Thomas, R. N. (1965) *Some Aspects of Non-Equilibrium Thermodynamics in the Presence of a Radiation Field*. Boulder: University of Colorado.

(U1) Unsöld, A. (1955) *Physik der Sternatmosphären*. (2nd ed.) Berlin: Springer.

(W1) Woolley, R. v.d. R. and Stibbs, D. W. N. (1953) *The Outer Layers of a Star*. Oxford: Oxford University Press.